Lecture Notes
The Lucas Island Model

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The Lucas (1972) island model is an overlapping generations model: at each period except the first N agents are born and they live for two periods. The agents have utility from consumption and leisure and maximize

$$u(c_t^t, n_t^t) + \mathbb{E}[\bar{u}(c_{t+1}^t)].$$

OLG model with only labor. Production is linear in labor $f(n) = n$.

Fiat money is given to the old generation proportionally to their current holding.

The only trade that occurs is money from the old to the young in exchange for goods. This is *not* an AD economy.

The old are split between islands to make total money demand equal in both islands. The young are split $\theta/2$ in one island and $1-\theta/2$ in the other. The initial money holding M is known, but $M' = xM$ is not known directly. θ is also not known.

Assume that $x \in \mathbb{R}_{++}$ are iid and so are $\theta \in (0, 2)$.

Utility and constraint are

$$u(c, n) + \mathbb{E}[\bar{u}(c')].$$

Constraints:

$$pc + m \leq wn, \quad p'c' \leq mx'.$$

Lagrangian

$$L = u(c, n) + \bar{u}(c') - \mu[pc - wn + p'c'/x']$$

From the firm's problem

$$\pi = (p-w)n \;\Rightarrow\; w = p.$$

constraint + FOCs

$$\begin{aligned}
&m = p(n-c)\\
&u_c(c,n) - \mu p = 0\\
&u_n(c,n) + \mu p = 0\\
&p'c' = mx'\\
&\mathbb{E}[\bar{u}_c(c') - \mu p'/x'] = 0.
\end{aligned}$$

Rewrite

$$\begin{aligned}
&m/p = n-c\\
&u_c(c,n) - \mu p = 0\\
&u_n(c,n) + \mu p = 0\\
&p'c' = mx'\\
&\mathbb{E}\left[\bar{u}_c\left(\frac{p'c'}{m}\right)\frac{x'}{p'}\middle|\mathfrak{F}_t\right] = \mu.
\end{aligned}$$

The first three can be combined to solve for $c, n, p\mu$ as a function of m/p.

For example if $u(c,n) = \log c + \sigma \log(1-n)$,

$$c = \frac{1}{1+\sigma}\left(1 - \frac{m}{p}\right), \qquad n = \frac{1}{1+\sigma}\left(\sigma\frac{m}{p} + 1\right), \qquad p\mu = \frac{1+\sigma}{1-m/p} \tag{1}$$

Define $h(m) = p\mu(m/p)$. [1] The remaining equation is

$$h\left(\frac{m}{p}\right)\frac{1}{p} = \mathbb{E}\left[\bar{u}_c\left(\frac{mx'}{p'}\right)\frac{x'}{p'}\middle|\mathfrak{F}_t\right]$$

The old sell all of their money to buy consumption goods, so the money supply in each island is $NMx/2$. The money supplied per young demander in the first island is $(NMx/2)/(N\theta/2) = Mx/\theta$. Market clearing for money is therefore $Mx/\theta = m$. Thus

$$h\left(\frac{Mx}{\theta p}\right)\frac{1}{p} = \mathbb{E}\left[\bar{u}_c\left(\frac{Mxx'}{\theta p'}\right)\frac{x'}{p'}\middle|p\right] \tag{2}$$

An equilibrium in this system is a price function $p(M, x, \theta)$ such that:

[1] For the example: $h(\xi) = (1+\sigma)/(1-\xi)$

1. All agents are acting optimally given that they believe that prices in every period are given by the price function.
2. Markets clear.

As long as the solution is interior, this boils down to the single equation (2). Lucas shows that the above always has a solution of the form $p(M, x, \theta) = M\phi(x/\theta)$, with ϕ positive and continuously differentiable.

1.1 A simple example

In the example (1), add the assumption that $\bar{u}(c) = 2\sqrt{c}$. The equation (2) becomes

$$\frac{1+\sigma}{\phi(x/\theta) - x/\theta} = \mathbb{E}\left[\sqrt{\frac{\theta x'}{x^2\phi(x'/\theta')}}\bigg| p\right]$$

Let us also assume that (x, x', θ, θ') are all independent of each other and that all are equal to either $2/3$ or $3/2$ with an equal probability of $1/2$. First of all, this allows us to separate the expectation to

$$\mathbb{E}\left[\sqrt{\frac{\theta x'}{x^2\phi(x'/\theta')}}\bigg| p\right] = \mathbb{E}\left[\sqrt{\frac{\theta}{x^2}}\bigg| p\right]\mathbb{E}\left[\sqrt{\frac{x'}{\phi(x'/\theta')}}\right].$$

We are going to guess that the price function fully reveals the value of x/θ, thus the first expectation is

$$\mathbb{E}\left[\sqrt{\frac{\theta}{x^2}}\bigg|\frac{x}{\theta}\right] = \sqrt{\frac{\theta}{x}}\,\mathbb{E}\left[x^{-1/2}\bigg|\frac{x}{\theta}\right] = \begin{cases} (3/2)^{3/2} & x/\theta = 4/9 \\ 5/\sqrt{6} & x/\theta = 1 \\ (2/3)^{3/2} & x/\theta = 9/4 \end{cases}$$

and the second is

$$\mathbb{E}\left[\sqrt{\frac{x'}{\phi(x'/\theta')}}\right] = \frac{1}{4\sqrt{6}}\left(\frac{5}{\sqrt{\phi(1)}} + \frac{3}{\sqrt{\phi(9/4)}} + \frac{2}{\sqrt{\phi(4/9)}}\right)$$

Thus, the set of equations that we need to solve is

$$\frac{1+\sigma}{\phi(4/9) - 4/9} = \frac{(3/2)^{3/2}}{4\sqrt{6}}\left(\frac{5}{\sqrt{\phi(1)}} + \frac{3}{\sqrt{\phi(9/4)}} + \frac{2}{\sqrt{\phi(4/9)}}\right),$$

$$\frac{1+\sigma}{\phi(1) - 1} = \frac{5/\sqrt{6}}{4\sqrt{6}}\left(\frac{5}{\sqrt{\phi(1)}} + \frac{3}{\sqrt{\phi(9/4)}} + \frac{2}{\sqrt{\phi(4/9)}}\right),$$

$$\frac{1+\sigma}{\phi(9/4) - 9/4} = \frac{(2/3)^{3/2}}{4\sqrt{6}}\left(\frac{5}{\sqrt{\phi(1)}} + \frac{3}{\sqrt{\phi(9/4)}} + \frac{2}{\sqrt{\phi(4/9)}}\right).$$

It is straightforward to show that this system always has a unique solution. What's interesting for us is that money is not neutral, e.g.

$$\frac{p(M, x=3/2, \theta=2/3)}{p(M, x=2/3, \theta=2/3)} = \frac{\phi(9/4)}{\phi(1)} > \frac{9}{4},$$

and

$$\frac{p(M, x=3/2, \theta=3/2)}{p(M, x=2/3, \theta=3/2)} = \frac{\phi(1)}{\phi(4/9)} < \frac{9}{4},$$

In other words, when the money supply increases by a factor of 9/4 prices do not rise by the same factor. In the island with the larger (smaller) population prices rise by a smaller (a larger) factor. This is because the increase in money supply is mistaken by the agents to some extent to be a change in the young population of the island.

1.2 Nonstochastic x

Consider the case when x is not stochastic and always equal to $\bar{x}$.

$$h\left(\frac{\bar{x}/\theta}{\phi(\bar{x}/\theta)}\right)\frac{1}{\phi(\bar{x}/\theta)} = \mathbb{E}\left[\bar{u}_c\left(\frac{\bar{x}/\theta}{\phi(\bar{x}/\theta')}\right)\frac{1}{\phi(\bar{x}/\theta')}\Big|\phi(\bar{x}/\theta)\right]$$

If ϕ is constant then the RHS is constant, which is impossible, so we assume that ϕ reveals θ, and the equation is

$$h\left(\frac{\bar{x}/\theta}{\phi(\bar{x}/\theta)}\right)\frac{1}{\phi(\bar{x}/\theta)} = \mathbb{E}\left[\bar{u}_c\left(\frac{\bar{x}/\theta}{\phi(\bar{x}/\theta')}\right)\frac{1}{\phi(\bar{x}/\theta')}\Big|\phi(\bar{x}/\theta)\right]$$

This proves that $\theta\phi$ must be constant, so $p = M\phi(\bar{x}/\theta) \propto M\bar{x}/\theta$.

1.3 Nonstochastic θ

Similarly, consider the case where the island population is constant at $\theta = 1$. Then

$$h\left(\frac{x}{\phi(x)}\right)\frac{1}{\phi(x)} = \mathbb{E}\left[\bar{u}_c\left(\frac{x'}{\phi(x')}\right)\frac{x'}{x\phi(x')}\Big|\phi(x)\right].$$

As before, ϕ cannot be constant, so it must reveal x, and therefore

$$h\left(\frac{x}{\phi(x)}\right)\frac{1}{\phi(x)} = \mathbb{E}\left[\bar{u}_c\left(\frac{x'}{\phi(x')}\right)\frac{x'}{x\phi(x')}\Big|\phi(x)\right].$$

Then

$$h\left(\frac{M\bar{x}}{\theta p}\right)\frac{1}{p} = \mathbb{E}\left[\bar{u}_c\left(\frac{M\bar{x}^2}{\theta p'}\right)\frac{\bar{x}}{p'}\right].$$

Let's add to the example that $\bar{u} = \beta \log c$.