Lecture Notes 7
More Stochastic Dynamic Programming

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1 Arrow-Debreu Securities

An Arrow-Debreu security is a contract that promises to pay one unit of a good at time t if the history up to that point is s^t. It is a useful way to think about trade in an economy with uncertainty: for example, if an agent has a stochastic endowment of a good $\{y_t(s^t)\}_{t=0}^{\infty}$, we can imagine the agent as sitting at time $t = 0$ with pieces of paper saying "this paper pays one unit of good at time t if the history is s^t". Trade is then a matter of trading these pieces of paper at $t = 0$, where, in the competitive case, there is a price $q^t(s^t)$ for each such security.

Modeling an economy using Arrow-Debreu securities makes some very strong implicit assumptions about the nature of uncertainty. For example, it requires that the state of the world can be verified by all agents, so that all agents can agree on whether or not a security is valid at every point in time. This is the assumption of complete markets, and while there are many reasons that it is not realistic, it is useful as a benchmark.

1.1 Example: An Endowment Economy

An economy is populated with consumers of different types indexed by $i = 1, 2, \dots, I$. All types have the same utility function, and maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i), \qquad u(c) = c^{1-\sigma}/(1-\sigma),\ \sigma > 0$$

There is a stochastic process denoted by $\{a_t\}$, and s^t denotes the history of the process up to time t. Each consumer is endowed with a stream of a nondurable good denoted by $\{y_t^i(s^t)\}_{t=0}^{\infty}$. We emphasize that the history s^t determines the endowments of *all* agents. Let $q_0^t(s^t)$ denote the price of the Arrow-Debreu (AD) security for one unit of the good at time t when the history is s^t. The budget constraint for consumer i is therefore

$$\sum_{t=0}^{\infty}\sum_{s^t} q_0^t(s^t)c_t^i(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_0^t(s^t)y_t^i(s^t).$$

The lagrangian for consumer i is

$$L^i = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i) - \lambda^i \sum_{t=0}^{\infty}\sum_{s^t} q_0^t(s^t)[c_t^i(s^t) - y_t^i(s^t)].$$

Thus, the FOCs are

$$\beta^t P(s^t|s^0)u'(c_t^i(s^t)) = \lambda^i q_0^t(s^t). \tag{1}$$

This gives a solution for $c_t^i(s^t)$ in terms of the prices and λ^i, and the latter can be determined by using the budget constraint.

Comparing two different agents:

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\lambda^i}{\lambda^j}.$$

Notice that the right-hand-side is independent of both t and s^t, implying that the ratio of the marginal utilities of consumption is the same between any two agents both at every period and at any realization of uncertainty. Specifically, for the CRRA we chose

$$\frac{c_t^i(s^t)}{c_t^j(s^t)} = \left(\frac{\lambda^j}{\lambda^i}\right)^{1/\sigma}. \tag{2}$$

To close the model, we add market clearing conditions for each t and s^t

$$\sum_i c_t^i(s^t) = \sum_i y_t^i(s^t).$$

Using (2) with $j = 1$ in the above,

$$c_t^1(s^t)(\lambda^1)^{1/\sigma}\sum_i(\lambda^i)^{-1/\sigma} = \sum_i y_t^i(s^t). \tag{3}$$

Notice that the only dependency on the state of the world comes through the aggregate endowment on the right-hand-side. In other words at each history s^t the consumption of consumers of type 1 depends only on the aggregate endowment of the economy in the current state, and obviously, since the choice $j = 1$ is arbitrary, this is true for all types.

If we further assume that the total endowment is the same in all states of the world, i.e. $\sum_i y_t^i(s^t) = \bar{y}_t$ for all s^t, then (3) also implies that c_t^1 independent of s^t: trade allows each agent to perfectly insure herself so that she consumes the same at all states of the world. Overall, this economy displays a high degree of consumption smoothing along a few dimensions: both across agents and across states.

In the case $\sigma = 1$ and $\bar{y}_t = \bar{y}$, we get that each agent consumes a constant amount at each date and at each realization of uncertainty given by:

$$c_t^i(s^t) = \bar{c}^i = (1-\beta)\sum_{\tau=0}^{\infty}\sum_{s^\tau}\beta^t P(s^t|s^0)y_t^j(s^\tau),$$

and the Arrow-Debreu prices are

$$q_0^t(s^t) = \beta^t P(s^t|s^0),$$

where we have normalized $q_0^t(s^0) = 1$.

2 Linearization of the RBC Model

The RBC model is given by

$$\begin{aligned} v(k,a) = &\max_{c,n,k'}\Big\{u(c,n) + \beta\mathbb{E}[v(k',a')|a]\Big\} \\ &\text{s.t.} \\ &c + k' \leq (1-\delta)k + e^a F(k,n) \\ &c \geq 0,\ k' \geq 0,\ n \in [0,1] \end{aligned}$$

The first order and envelope conditions can be combined to give:

$$u_c(c,n) = \beta\mathbb{E}\Big[u_c(c',n')(1-\delta+e^{a'}F_k(k',n'))|a\Big], \tag{4a}$$

$$\frac{-u_n(c,n)}{u_c(c,n)} = e^a F_n(k,n) \tag{4b}$$

$$c + k' = (1-\delta)k + e^a F(k,n). \tag{4c}$$

These are referred to as the inter-temporal equation, intra-temporal equation, and resource constraint, respectively.

Let us assume that a_t is an AR(1) process, i.e.

$$a_{t+1} = \rho a_t + \varepsilon_t,$$

where $\{\varepsilon_t\}$ are independent and are identically distributed $\mathrm{N}(0, \sigma^2)$. The system (4) can be written as a vector equation

$$\mathbb{E}[G(c, n, k, c', n', k', a, \varepsilon; \sigma)|a] = 0.$$

Linearization in the stochastic case is an expansion around a deterministic steady-state, which is the same as saying $\varepsilon = 0$ and $a_0 = 0$. It can be viewed as expanding around known solution:

$$G(c^*, n^*, k^*, c^*, n^*, k^*, 0, \varepsilon; 0),$$

for initial conditions close to the steady state and a small σ.

2.1 The Steady State

Finding the steady state is the same as in the deterministic case: we need to solve

$$1 = \beta(1 - \delta + F_k(k^*, n^*)), \tag{5a}$$

$$\frac{-u_n(c^*, n^*)}{u_c(c^*, n^*)} = F_n(k^*, n^*) \tag{5b}$$

$$c^* = F(k^*, n^*) - \delta k^*. \tag{5c}$$

For the sake of concreteness, we choose the following specification:

$$u(c, n) = \log c + \psi \log(1 - n),$$
$$F(k, n) = k^\alpha n^{1-\alpha}.$$

The system (5) becomes

$$1 = \beta(1 - \delta + \alpha k^{*\alpha-1} n^{*1-\alpha}), \tag{6a}$$

$$\psi \frac{c^*}{1 - n^*} = (1 - \alpha) k^{*\alpha} n^{*-\alpha} \tag{6b}$$

$$c^* = k^{*\alpha} n^{*1-\alpha} - \delta k^*. \tag{6c}$$

Equation (6a) determines k^*/n^* in terms of the parameters α, β and δ:

$$\left(\frac{k^*}{n^*}\right)^{1-\alpha} = \frac{\alpha}{\beta^{-1} - 1 + \delta},$$

and this can be used in (6c) to determine c^*/k^*:

$$\frac{c^*}{k^*} = \frac{\beta^{-1} - 1 + (1 - \alpha)\delta}{\alpha}.$$

Choosing the parameter values $\alpha = 1/3, \delta = 0.1$ and $\beta^{-1} = 1.03$, we get $k^*/n^* \approx 4.1$ and $c^*/k^* = 0.29$. Rather than solving (6b) for n^*, a standard trick is to require $n^* = 1/3$ (assuming that people work about 8 hours a day) and to solve for ψ. With the values we chose, this gives $\psi \approx 1.8$. In addition, we set $\rho = 0.82$.

2.2 Linearized Equations

The next step is to linearize the system (4) around the steady state that we found:

$$\hat{c} = \mathbb{E}\Big[\hat{c}' + (1-\alpha)(1-\beta(1-\delta))(\hat{k}' - \hat{n}') - [1-\beta(1-\delta)]a'\Big] \tag{7a}$$

$$\hat{c} + \left(\frac{n^*}{1-n^*} + \alpha\right)\hat{n} - \alpha\hat{k} - a = 0 \tag{7b}$$

$$\frac{c^*}{k^*}\hat{c} + \hat{k}' - \beta^{-1}\hat{k} - (1-\alpha)\left(\frac{c^*}{k^*} + \delta\right)\hat{n} - \left(\frac{c^*}{k^*} + \delta\right)a = 0 \tag{7c}$$

where

$$\hat{c} = \frac{c-c^*}{c^*}, \qquad \hat{n} = \frac{n-n^*}{n^*}, \qquad \hat{k} = \frac{k-k^*}{k^*}.$$

We can rewrite the system (7) in matrix notation

$$0 = \mathbb{E}[Ax + Bx' + Qa + R\varepsilon' | a],$$

where, with the parameter values chosen,

$$x = (\hat{c}, \hat{k}, \hat{n})^T,$$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1/3 & 5/6 \\ 0.29 & -1.03 & -0.26 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0.0841 & -0.0841 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} -0.1035 \\ -1 \\ -0.39 \end{pmatrix}, R = \begin{pmatrix} -0.1262 \\ 0 \\ 0 \end{pmatrix}.$$

We have used the fact that $a' = \rho a + \varepsilon$.

The matrix B is not invertible since the second row has only zeros in it. This is simply due to the fact that the intra-temporal equation (7b) does not mix variables from different periods. We can thus use (7b) to eliminate the $\hat{c}$, and use this to reduce to order of the system to two. We find:

$$0 = \mathbb{E}[Ax + Bx' + Qa + R\varepsilon' | a],$$

where now

$$x = (\hat{k}, \hat{n})^T,$$

$$A = \begin{pmatrix} -1/3 & 5/6 \\ -0.9333 & -0.5017 \end{pmatrix}, \; B = \begin{pmatrix} 0.4175 & -0.9175 \\ 1 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} -0.2835 \\ -0.1 \end{pmatrix}, R = \begin{pmatrix} 0.8738 \\ 0 \end{pmatrix}.$$

Notice that

$$\begin{aligned}
&0 = \mathbb{E}[Ax + Bx' + Qa + R\varepsilon'|a] = B\mathbb{E}[x'] + Ax + Qa,\\
&B\mathbb{E}[x'|a] = -Ax - Qa,\\
&\mathbb{E}[x'|a] = (-B^{-1}A)x + (-B^{-1}Q)a.
\end{aligned}$$

Just as we did in the deterministic case, let $-B^{-1}A = V\Lambda V^{-1}$, where $\det V \neq 0$ and Λ is diagonal.[1] Also, let $x_t = Vw_t$. We have

$$\begin{aligned}
&V\mathbb{E}[w_{t+1}|a] = V\Lambda w_t + (-B^{-1}Q)a_t,\\
&\mathbb{E}[w_{t+1}|a] = \Lambda w_t + Ta_t,
\end{aligned}$$

where $T = -V^{-1}B^{-1}Q$.

We calculate:

$$\begin{aligned}
\mathbb{E}_0[w_t] &= \mathbb{E}_0[\mathbb{E}_{t-1}[w_t]] = \mathbb{E}_0[\Lambda w_{t-1} + Ta_{t-1}]\\
&= \mathbb{E}_0[\mathbb{E}_{t-2}[\Lambda w_{t-1} + Ta_{t-1}]] = \mathbb{E}_0[\Lambda^2 w_{t-2} + \Lambda T a_{t-2} + Ta_{t-1}]\\
&= \cdots = \Lambda^t w_0 + \sum_{s=0}^{t-1} \Lambda^{t-s-1} T a_s
\end{aligned}$$

In the deterministic case, an equation similar to the one above demonstrated that w_0 must be chosen such that the elements corresponding to eigenvalues outside the unit circle must vanish. In the stochastic case we get a similar condition: we require the same of $\Lambda w_t + Ta_t$, i.e. out decision rule $\hat{n}(k,a)$ is such that $\Lambda w_t + Ta_t$ remain in the subspace associated with the eigenvalues that are within the unit circle.

Calculating this explicitly, we find

$$\begin{aligned}
&\Lambda = \begin{pmatrix} 0.832 & 0 \\ 0 & 1.238 \end{pmatrix}, \quad V = \begin{pmatrix} -0.980 & -0.855 \\ 0.198 & -0.519 \end{pmatrix}, \quad T = \begin{pmatrix} -0.409 \\ 0.352 \end{pmatrix}.\\
&\Lambda V^{-1} \begin{pmatrix} \hat{k} \\ \hat{n} \end{pmatrix} + Ta = \begin{pmatrix} \cdots \\ -0.3611\hat{k} - 1.7909\hat{n} + 0.3521a \end{pmatrix}\\
&-0.3611\hat{k} - 1.7909\hat{n} + 0.3521a = 0\\
&\hat{n} = -0.2016\hat{k} + 0.1966a.
\end{aligned}$$

2.3 Impulse Response Functions

Now that we know the decision rule for $\hat{n}$, we can back out the decision rules for $\hat{c}$ and $\hat{k}'$. We can then, of course, also calculate changes to output, the interest rate,

[1]More precisely, Λ is the Jordan form of $-B^{-1}A$

wages and so on. A convenient way of displaying these results is using *impulse response functions* (IRF). This is fancy term for plotting the trajectory of all variables of interest to a specific realization of uncertainty: the economy starts at steady state and is hit by a shock at time $t = 0$. Formally,

$$\varepsilon_t = \begin{cases} \sigma & t = 0 \\ 0 & t \neq 0 \end{cases},$$

for some specific σ, and $k_0 = k^*$. The IRFs display how the economy evolves in response to the shock.

Figure 1 presents IRFs for $\sigma = -0.027$. This value is chosen to create a 3% drop in output at $t = 0$, as one might observe in a severe crisis.

3 General Linearization

Returning to the general case, whenever we solve a dynamic programming problem, we end up with a set of equations that can be summarized as

$$\mathbb{E}_t[G(x, x', x'', a, a')] = 0.$$

Let $x \in \mathbb{R}^l$, and assume that $a \in \mathbb{R}^k$ is a vector-AR(1) process:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1}, \qquad \varepsilon_t \sim \mathrm{N}(0, \sigma^2 I_k),$$

where $\varepsilon \in \mathbb{R}^k$, and ρ is k-by-k.

Suppose that we know a solution for $x_t = x^*$, $\sigma = 0$. Expanding to first order around that solution

$$\mathbb{E}_t[P_0 x + P_1 x' + P_2 x'' + Q_0 a + Q_1 a'] = 0,$$

where P_i are l-by-l and Q_i are l-by-k. In matrix notation

$$\mathbb{E}_t \begin{pmatrix} x_{t+2} \\ x_{t+1} \\ a_{t+1} \end{pmatrix} = - \begin{pmatrix} P_2 & 0 & Q_1 \\ 0 & I_l & 0 \\ 0 & 0 & I_k \end{pmatrix}^{-1} \begin{pmatrix} P_1 & P_0 & Q_0 \\ -I_l & 0 & 0 \\ 0 & 0 & -\rho \end{pmatrix} \begin{pmatrix} x_{t+1} \\ x_t \\ a_t \end{pmatrix}$$

As before, what we must do is diagonalize the $2l + k$ dimensional matrix above and calculate its eigenvalues. Denote the number of eigenvalues outside the unit circle as m. We have l degrees of freedom to choose in the vector (x_{t+1}, x_t, a_t), which means that we can distinguish between three cases:

Case 1 Determinacy: $m = l$. Here we have exactly enough degrees of freedom to uniquely determine the linear solution.

Case 2 Instability: $m > l$**.** We do not have enough degrees of freedom, so there is no convergence to the steady-state.

Case 3 Indeterminacy: $m < l$**.** Here we have more degrees of freedom than we need, which means that there is more than one solution.

3.1 Sunspots

William Stanley Jevons (1835-1882) was one of the economists responsible for the marginal revolution, which was the first attempt to build economic theory on mathematical foundations. In his late life one of the ideas that he was studying was that sunspots (on the actual sun) affect the business cycle through influencing the price of corn.

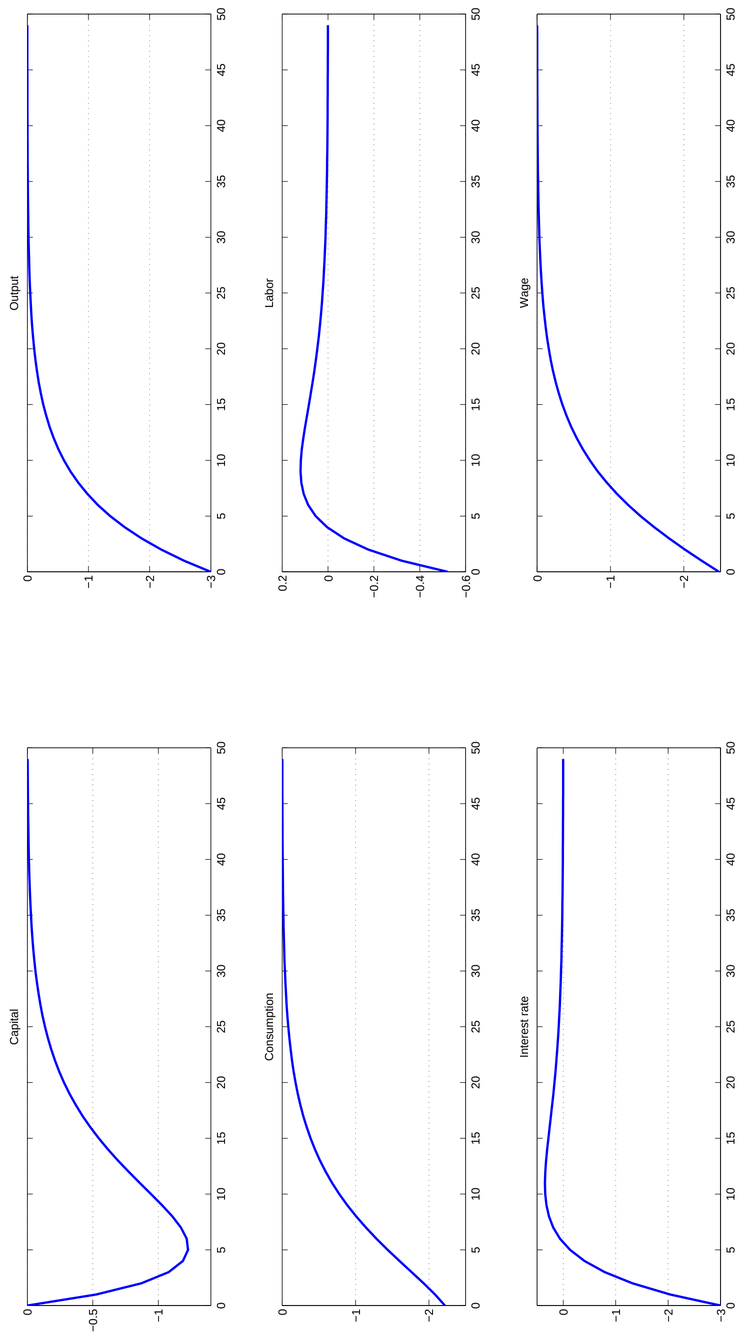

Figure 1: Impulse response functions for the RBC model. The horizontal access is percentage change.