# Lecture Notes 5
# More on Growth

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## 1 Growth through specialization

In this section we shall discuss a model similar to Romer's 1987 paper.[1] Based on the Dixit-Stiglitz math, which we discussed last week, assume that the final good is produced competitively using labor and intermediate goods according to

$$Y_t = L_t^{1-\alpha} \int_0^{A_t} x_t(i)^\alpha di,$$

where $A_t > 0$ represents the number of varieties that can be produced at time $t$, $L$ is labor, and $x_t(i)$ are the intermediate goods. The intermediate goods are produced using capital alone according to $x_t(i) = k_t(i)$ for $i \in [0, A_t]$.

Notice that if, for example, all firms had the same amount of capital $k_t$ (as will be the case in equilibrium), then production is simply $Y = A_t L_t^{1-\alpha} k_t^\alpha$, i.e. the standard Cobb-Douglas production function. Thus, the production function exhibits increasing returns to scale in the three inputs that can be chosen by the economy: $k$, $L$ and $A$, but we can have a competitive equilibrium nonetheless, because different inputs are determined by separate firms, and due to the monopolistic power that the intermediate good producers have.

As in previous models, output at time $t$ can be used to for consumption or for accumulating capital for the next period. In addition, output can be used can for research and development, which increases the number of varieties producible with a constant marginal return $\kappa$. That is, every $\kappa$ units of output can be used to increase $A_{t+1}$ by one unit.

[1] Romer, Paul M. "Growth Based on Increasing Returns Due to Specialization." *The American Economic Review* 77, no. 2 (May 1, 1987): 5662.

## 1.1 Monopolistic Competition

In the decentralized version of this model there are households who maximize their utility, for whom FOCs and constraint are:

$$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = \beta[1-\delta+r_{t+1}], \tag{1}$$

$$L_t = 1, \tag{2}$$

$$c_t + k_{t+1} = (1-\delta)k_t + w_t L + r_t k_t + \Pi_t, \tag{3}$$

where $\Pi_t$ is other income that will come from the total profits of all the firms.

The final good firms' FOCs define the demand functions for labor and for the intermediate goods:

$$w_t = (1-\alpha)L_t^{-\alpha}\int_0^{A_t} x_t(i)^{\alpha}di = (1-\alpha)\frac{Y_t}{L_t},$$

$$p_t(i) = \alpha L_t^{1-\alpha}x_t(i)^{\alpha-1}$$

The monopolisitc intermediate good producers maximize $\pi_t(i) = [p_t(i) - r_t]x_t(i)$, which gives the equations

$$r_t = \alpha^2 L_t^{1-\alpha}x_t(i)^{\alpha-1} = \alpha p_t(i), \tag{4}$$

$$\pi_t(i) = \alpha(1-\alpha)L_t^{1-\alpha}x_t(i)^{\alpha} = (1-\alpha)\alpha^{(1+\alpha)/(1-\alpha)}L_t r_t^{-\alpha/(1-\alpha)} \tag{5}$$

In order to enter a market, a firm has to 'invent' a new good at cost $\kappa$. We shall assume that the firm then gets a patent granting it exclusive rights to produce that good forever. The entry condition dictates that the value of the patent, calculated in terms of a flow of profits discounted by the interest rate, is equal to the cost of producing the patent:

$$\kappa = \sum_{t=1}^{\infty}\pi_t(i)\prod_{s=1}^{t}(1+r_s)^{-1}. \tag{6}$$

We now have all we nned to solve the model. Specifically, we search for a solution with, $x_t(i) = x_m$ (constant) and the interest rate being constant $r_t = r_m$. Using this in (5) together with $L_t = 1$, and substituting into (6) gives $r$ in terms of the model parameters:

$$r_m = (1-\alpha)^{1-\alpha}\alpha^{1+\alpha}\kappa^{\alpha-1}.$$

Using this in (4)

$$x_m = \frac{\alpha\kappa}{1-\alpha}.$$

The rest of the equations then give expressions for capital, wages, consumption and technology. Notice that $w_t, A_t$ and $C_t$ all grow at the same rate for this solution. The growth rate is given by (1)

$$\mu^\sigma = \beta[1 - \delta + r_m].$$

## 1.2 Central Planner

The central planner faces the following economy-wide resource constraint

$$c_t + k_{t+1} + \kappa(A_{t+1} - A_t) = (1-\delta)k_t + L^{1-\alpha} \int_0^{A_t} x_t(i)^\alpha di.$$

Since the Dixit-Stiglitz production function is concave and symmetric in the intermediate good inputs, the central planner will always find it optimal to spread the capital equally between all of the intermediate-good producers that exist:

$$x_t(i) = \frac{k_t}{A_t} \qquad \Rightarrow \qquad \int_0^{A_t} x_t(i)^\alpha di = A_t^{1-\alpha} k_t^\alpha,$$

so we can rewrite

$$c_t + k_{t+1} + \kappa(A_{t+1} - A_t) = (1-\delta)k_t + A_t^{1-\alpha} L^{1-\alpha} k_t^\alpha.$$

The lagrangian for the central planner is therefore

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) - \lambda_t \left[ c_t + k_{t+1} + \kappa(A_{t+1} - A_t) - (1-\delta)k_t - A_t^{1-\alpha} L_t^{1-\alpha} k_t^\alpha \right] \right\},$$

so the FOCs are

$$\begin{aligned} & L_t = 1, \\ & u'(c_t) = \lambda_t, \\ & \lambda_t = \beta \lambda_{t+1} \left[ 1 - \delta + \alpha A_{t+1}^{1-\alpha} L_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} \right], \\ & \lambda_t \kappa = \beta \lambda_{t+1} \left[ \kappa + (1-\alpha) A_{t+1}^{-\alpha} L_{t+1}^{1-\alpha} k_{t+1}^{\alpha} \right]. \end{aligned}$$

Combining the first three equations gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta[1 - \delta + r_{c,t+1}], \tag{7}$$

where

$$r_{c,t} = \alpha A_t^{1-\alpha} k_t^{\alpha-1}$$

is the shadow interest rate, which is equal to the marginal product of capital. Of course, there is no actual interest rate in this economy because there are no markets in a centrally planned economy, but the planner saves as if the interest rate is $r_c$. Comparing (1) and (7) also allows to compare growth rates directly by comparing $r_c$ to $r_m$.

We are searching for a constant growth path, so the last equation tells us that $k_t$ and $A_t$ are growing at the same rate, and that $r_c = \alpha(A/k)^{*1-\alpha}$.

Combining the last two equations gives

$$\frac{1}{\kappa} = \frac{1 - \delta + \alpha(A/k)^{*1-\alpha}}{\kappa + (1-\alpha)(A/k)^{*-\alpha}}.$$

Expressing this in terms of $r_c$ yields an equation for the steady state interest rate (and, thus, indirectly for the rate of growth):

$$\kappa^{-1}(1-\alpha)\alpha^{\alpha/(1-\alpha)}r_c^{-\alpha/(1-\alpha)} - r_c + \delta = 0.$$

## 1.3 Comparison

First of all, we wish to prove that $r_c > r_m$, which is equivalent to the statement that the growth rate in the centralized economy is higher. Define

$$G_c(r) = \kappa^{-1}(1-\alpha)\alpha^{\alpha/(1-\alpha)}r^{-\alpha/(1-\alpha)} - r + \delta.$$

Note that $G_c(0) = +\infty$, $G_c(+\infty) = -\infty$, and $G_c'(r) < 0$, therefore $G_c(r)$ has exactly one root at $r = r_c$. Next define $\tilde{G}(r) = G_c(r) - \delta$. Since $\tilde{G}(r) < G_c(r)$ for all $r$, the roots of $\tilde{G}$ are all smaller than the roots of $G_c$, but $\tilde{G}$ has only one root at

$$\tilde{r} = \kappa^{\alpha-1}(1-\alpha)^{1-\alpha}\alpha^{\alpha} = r_m/\alpha.$$

Thus, $r_c > \tilde{r} = r_m/\alpha > r_m$.

We actually proved a stronger statement that we set out to prove: we showed that $r_m < \alpha r_c$. Why is this important? Due to monopoly power production is inefficiently low and capital is underutilized. We have seen before that this is manifested in the interest rate being a factor $\alpha$ lower than its efficient level. However, we get that the efficient growth rate is higher by more than a factor of $\alpha$, since there is an additional distortion: the patent system, which grants inventors all the profits from their invention, creates insufficient incentives for further R&D.

## 2 Growth Through Cycles

We have so far seen models where growth is attributed to accumulation of the factors of production, and models where growth is due to innovation. In a paper from 1999, Matsuyama demonstrates that it is entirely possible that the growth process cycles between these two modes of growth.[2] His model is based on Romer (1987) with one important difference: the patent that innovators get is only valid for a finite amount of time (one period in the model). The result is that, unlike the Romer model, it is not always profitable to innovate. In particular, when there is too little capital per firm, the returns to starting up a new firm are insufficient to incentivize innovation and the economy grows as in the neoclassical model, which Matsuyama refers to as the "Solow regime".

The analysis is similar to that of Romer's model, and the conclusions can be summarized by the policy function $(k/A)_{t+1} = \Phi((k/A)_t)$, which are depicted in the three graphs at the end of these notes. The figures are taken from the paper and $k$ in the original notation is what we call $k/A$. Matsuyama shows that depending on the parameters of the model, there are three cases: (1) the economy converges to a path where growth happens through factor accumulation as depicted in figure **??**, (2) the economy cycles between periods of accumulating capital and innovation as depicted in figure **??**, and, (3) the economy grows as in the Romer model above as depicted in figure **??**.

Turning attention to the second case, there are a few things worth noticing: first, there is a unique steady state $k^{**}$, but it is unstable. Second, while the economy does not converge to the steady state, it does remain forever at a finite distance from it, and follows a path that cycles between the two regimes.

## 3 The Overlapping Generations Model

Overlapping generations models are very useful for a variety of circumstances in macroeconomics. In these models a 'generation' of households is born at the beginning of every period and lives for two periods, except for one generation of 'old' households that exists when time starts. Formally, we define time to be discrete and runs from zero, i.e. $t = 0, 1, \ldots$, and allow an infinite set of types of agents marked by $i = -1, 0, 1, \ldots$, where agents of type $i$ have preferences that are given by the utility function

$$U_i = \begin{cases} u(c_i^i) + \beta u(c_{i+1}^i) & i \geq 0 \\ u(c_0^{-1}) & i = -1 \end{cases},$$

[2]Matsuyama, Kiminori. "Growing Through Cycles." *Econometrica* 67, no. 2 (1999): 335347.

where $\beta \in (0,1)$ is a discount factor and $c_j^i$ denotes the period $j$ consumption of the generation born at $i$.

In the version we study, only the young are endowed with one unit of leisure that they can sell for a wage $w_t$, and capital, as in the neoclassical growth model, is owned by households and rented out to firms. Notice that at the beginning of every period only the current period's old own capital and that at then end of the period only the young do.

Household born in period $t \geq 0$ solve

$$\max_{c_t^t, c_{t+1}^t, n_t, k_{t+1}} [u(c_t^t) + \beta u(c_{t+1}^t)], \tag{8}$$

subject to

$$\begin{aligned} &c_t^t + k_{t+1} \leq n_t w_t, \\ &c_{t+1}^t \leq (1-\delta + r_{t+1}) k_{t+1}, \\ &n_t \in [0,1],\, c_t^t, c_{t+1}^t, k_{t+1} \geq 0. \end{aligned}$$

The generation $i = -1$ maximizes $u(c_0^{-1})$ subject to $0 \leq c_0^{-1} \leq (1-\delta + r_0)k_0$.

There are also profit maximizing firms that use a constant-returns-to-scale production function $y = F(k,n) = nf(k/n)$. The firms' program is

$$\max_{k_t, n_t} [F(k_t, n_t) - r_t k_t - w_t n_t], \tag{9}$$

subject to $k_t, n_t \geq 0$.

Finally, feasibility requires

$$c_t^{t-1} + c_t^t + k_{t+1} \leq (1-\delta) k_t + F(k_t, n_t). \tag{10}$$

**Definition 1.** A **sequence of markets equilibrium** is a set of values for consumption, labor and capital, $\{c_0^{-1}, (c_t^t, c_{t+1}^t, k_{t+1}, n_t)_{t=0}^{\infty}\}$, prices $(w_t, r_t)_{t=0}^{\infty}$, such that for a given $k_0 \geq 0$:

1. All generations of households solve their maximization problem (8),
2. Firms maximize profits (9),
3. Prices are such that markets clear (10).

Solving the household's problem for all $t \geq 0$ leads to

$$\begin{aligned} &n_t = 1, \\ &\frac{u'(c_t^t)}{u'(c_{t+1}^t)} = \beta(1-\delta + r_{t+1}), \\ &c_t^t + k_{t+1} = c_t^t + \frac{c_{t+1}^t}{1-\delta + r_{t+1}} = w_t. \end{aligned}$$

The first generation adds

$$c_0^{-1} = (1 - \delta + r_0)k_0.$$

The firm's problem gives

$$\begin{aligned} r_t &= F_k(k_t, 1) = f'(k_t), \\ w_t &= F_n(k_t, 1) = f(k_t) - k_t f'(k_t). \end{aligned}$$

A solution to the last six equations is an equilibrium together with

$$c_t^{t-1} + c_t^t + k_{t+1} = (1 - \delta)k_t + f(k_t).$$

**Definition 2.** An equilibrium of the the OLG model is **stationary** if for all $t \geq 0$, $c_t^t = c^y$ and $c_{t+1}^t = c^o$.[3]

The Euler equation for a stationary equilibrium is

$$\frac{u'(c^y)}{u'(c^0)} = \beta(1 - \delta + r_{t+1}),$$

so $r_t$ must also be constant, and since $r_t = f'(k_t)$, so must $k_t$, which in turn implies that $w_t$ is constant. Overall, we get the following set of equations:

$$\begin{aligned} \frac{u'(c^y)}{u'(c^o)} &= \beta(1 - \delta + f'(k)), \\ c^y + k &= f(k) - kf'(k) \\ c^o + c^y &= f(k) - \delta k. \end{aligned}$$

This can be combined into

$$\frac{u'[f(k) - (1 + f'(k))k]}{u'[(1 - \delta + f'(k))k]} = \beta(1 - \delta + f'(k)).$$

However, note that it is not guaranteed that this equation has a solution.

[3]Notice that we do not require $c_0^{-1} = c^o$, so an equilibrium is still called stationary if the first generation is treated differently.