Lecture 3
Dynamic Planning: examples and dynamics

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1 Stability

In many applications in economics we find ourselves with an iterative mapping of the form:

$$x_{t+1} = f(x_t).$$

A *fixed point of* f is a point x^* such that $f(x^*) = x^*$. Obviously, if at some point in the sequence $x_t = x^*$ it will remain at x^* for all future t.

We are often interested in the question of whether or not the sequence will converge to x^* at least for x_0 close enough to x^*. This property is related to *stability*, and we will define it more accurately later in the course. However, for now, we mention some sufficient conditions. If $x \in \mathbb{R}$ and f is sufficiently smooth, then close enough to x^*

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*) = x^* + f'(x^*)(x - x^*),$$

so a sufficient condition is that $|f'(x^*)| < 1$.

If $x \in \mathbb{R}^n$ and $x_{j,t+1} = f_j(x_t)$, then the analogous condition is that all the eigenvalues of the matrix $(\partial f_j/\partial x_i)$ are all smaller than unity.

2 Example: Monopolistic Competition with Adjustment Costs

Consider a monopolistic firm that faces an inverse demand schedule of $p = a_0 - a_1 q$. Production is costless, but adjusting output costs $d \cdot (q_t - q_{t-1})^2/2$. The firm's period profits is thus $\pi_t = (a_0 - a_1 q_t)q_t - d \cdot (q_t - q_{t-1})^2/2$. The firm wishes to maximize the discounted sum of its profits:

$$U = \sum_{t=0}^{\infty} \beta^t \pi_t = \sum_{t=0}^{\infty} \beta^t \left[(a_0 - a_1 q_t) q_t - \frac{d}{2} \cdot (q_t - q_{t-1})^2 \right].$$

The Euler equation is

$$\beta d q_{t+2} - (2a_1 + d + \beta d) q_{t+1} + d q_t = 0.$$

This is a linear second order difference equation. To find the general solution we solve the homogenous equation:

$$\beta d q_{t+2} - (2a_1 + d + \beta d) q_{t+1} + d q_t = -a_0,$$

by guessing $q(t) = \lambda^t$:

$$\beta d \lambda_2 - (2a_1 + d + \beta d)\lambda + d = 0 \Rightarrow \lambda_{\pm} = \frac{2a_1 + d + \beta d \pm \sqrt{(2a_1 + d + \beta d)^2 - 4\beta d^2}}{2\beta d}.$$

We find the special solution by guessing $q_t = q^*$ and finding

$$\beta d q^* - (2a_1 + d + \beta d) q^* + d q^* = -a_0 \Rightarrow q^* = \frac{a_0}{2a_1}.$$

The general solution is therefore

$$q_t = \frac{a_0}{2a_1} + C_+ \lambda_+^t + C_- \lambda_-^t.$$

Recall that the product of two roots of a quadratic equation is $\gamma_0/\gamma_2 = 1/\beta > 1$, so both roots are of the same sign. Furthermore, the sum is $\gamma_1/\gamma_0 = 1 + 1/\beta + 2a_1/(\beta d) > 2$, so both roots are positive and at least one of them is larger than one. It is in fact not difficult to verify that $\lambda_+ > 1$ and $0 < \lambda_- < 1$. Since the solution must not diverge, we must have $C_+ = 0$. Finally substituting $t = 0$ in the general solution, we find $C_- = q_0 - a_0/(2a_1)$, therefore the solution is

$$q_t = \frac{a_0}{2a_1} + \left(q_0 - \frac{a_0}{2a_1} \right) \lambda_-^t,$$

and the implied policy function is

$$q_{t+1} = \frac{a_0}{2a_1} + \left(q_t - \frac{a_0}{2a_1} \right) \lambda_-. \quad (1)$$

Notice that q_t approaches $q^* = a_0/(2a_1)$, which is the optimal choice for a monopolist. The solution we found describes how quickly the firm approaches q^* when faced with adjustment costs.

Turning to the recursive formalism, the functional equation for this problem is

$$v(q) = \max_{q'} \left[(a_0 - a_1 q')q' - \frac{d}{2}(q' - q)^2 + \beta v(q') \right].$$

The FOC is

$$a_0 - 2a_1 q' - d(q' - q) + \beta v'(q') = 0,$$

and the envelope equation is

$$v'(q) = d(q' - q).$$

The form of these equations suggests that a quadratic value function is appropriate. Furthermore, v should be maximized when $q = q^*$, so we guess $v(q) = w_2 q(q - 2q^*) + w_0$. Substituting this into the envelope condition gives

$$2w_2(q - q^*) = d(q' - q) \quad \Rightarrow \quad q' = q\left(\frac{2w_2}{d} + 1\right) - \frac{a_0 w_2}{a_1 d} \tag{2}$$

Using this in the FOC gives the linear equation:

$$\left[a_1 d + (2a_1 + (1 - \beta)d)w_2 - 2\beta w_2^2\right]\left(\frac{a_0}{a_1} - 2q\right) = 0$$

Since this must be true for all q, the term in the brackets must vanish identically. It follows that

$$w_2 = \frac{2a_1 + (1 - \beta)d \pm \sqrt{(2a_1 + (1 - \beta)d)^2 + 8\beta d a_1}}{4\beta}. \tag{3}$$

Since w_2 must be negative, we can ignore the solution with the + sign. Substituting the negative solution into (2) gives us the same policy function we found in (1).

Notice that w_0 doesn't enter anywhere in the calculation, so there is no real need to find it. However, if you are curious, you can use the fact that if $q_0 = q^*$, then $q_t = q^*$ for all t, therefore

$$v(q^*) = \frac{1}{1 - \beta}(a_0 - a_1 q^*)q^* = \frac{1}{1 - \beta} \cdot \frac{a_0^2}{4a_1}.$$

2.1 Value function iteration

One method of solving the function equation consists of making an initial guess $v_0(x)$ and continuing by calculating:

$$v_{j+1}(x) = \max_{y\in\Gamma(x)}[F(x,y) + \beta v_j(y)].$$

The advantage of this method is that the sequence is guarantee to converge under conditions that are not too restrictive. The disadvantage is that the above equation is often difficult to calculate even numerically. We shall demonstrate this method for the adjustment costs example.

Let's start with $v_0(q) = 0$. Then

$$v_1(q) = \max_{q'}\left[(a_0 - a_1 q')q' - \frac{d}{2}(q'-q)^2\right].$$

Finding the solution to the FOC and substituting it back in, we find

$$v_1(q) = \frac{1}{4a_1 + 2d}[a_0^2 + 2a_0 dq - 2a_1 dq^2].$$

More generally, if at some point we reach a quadratic value function $v_j(q) = r_0^j + r_1^j q + r_2^j q^2$, then we need to solve

$$v_{j+1}(q) = \max_{q'}\left[(a_0 - a_1 q')q' - \frac{d}{2}(q'-q)^2 + \beta(r_0^j + r_1^j q + r_2^j q^2)\right]. \tag{4}$$

Solving the FOCs gives

$$q' = \frac{a_0 + \beta r_j^1 + dq}{2a_1 + d - 2\beta r_2^j},$$

and substituting this into the right hand side of (4) gives $v_{j+1}(q) = r_0^{j+1} + r_1^{j+1} q + r_2^{j+1} q^2$, where

$$r_2^{j+1} = \frac{d(\beta r_2^j - a_1)}{2a_1 + d - 2\beta r_2^j}, \tag{5a}$$

$$r_1^{j+1} = \frac{d(\beta r_1^j + a_0)}{2a_1 + d - 2\beta r_2^j}, \tag{5b}$$

$$r_0^{j+1} = \frac{(\beta r_1^j + a_0)^2}{2(2a_1 + d - 2\beta r_2^j)} + \beta r_0^j. \tag{5c}$$

To understand how this sequence converges, let us start with (5a). First note that r_2^{j+1} depends only on r_2^j, so we can write $r_2^{j+1} = \rho(r_2^j)$. Furthermore, this mapping has a fixed point exactly at the value of $r_2 = w_2$ that we found in (3), i.e. $\rho(w_2) = w_2$. At the fixed point:

$$\rho'(w_2) = \frac{4\beta d^2}{\left(2a_1 + (1+\beta)d + \sqrt{(2a_1 + (1-\beta)d)^2 + 8a_1 d\beta}\right)^2} < \beta < 1.$$

Thus, at least at some vicinity of w_2 the series r_2^j will converge to the fixed point.

The mapping in (5b) is linear in r_1^j. For r_2^j close enough to w_2 the slope is smaller than unity, so this mapping also converges. Finally, The mapping in (5c) is linear in r_0^j. So since r_1^j and r_2^j converge, so will r_0^j. The fixed points of these mappings are the correct value function for this problem.

3 Dynamics

So far, we have seen two methods of finding the value function: the "guess and verify" and the value function iteration. We will study more solution methods later, but before getting to that, it is important to know that there are often properties of the dynamics of the problem that can be proved without actually finding the value function or having a solution at all. Unfortunately, there are no general rules here, just tricks that may or may not work for any specific example. Therefore, we show one example of where we can say a lot (the neoclassical growth model) and one where the results don't hold.

3.1 Dynamics of the Neoclassical Growth Model

Recall the FOC, envelope condition, and resource constraint for the neoclassical growth model

$$u'(c) = \beta v'(k'),$$
$$v'(k) = [1 - \delta + f'(k)]u'(c),$$
$$c + k' = (1-\delta)k + f(k).$$

The assumptions on the production function

$$f(0) = 0, \qquad \lim_{k\to 0} f'(k) = +\infty, \qquad \lim_{k\to\infty} f'(k) = 0,$$

imply that there is some value $\bar{k}$ such that $f(\bar{k}) = \delta\bar{k}$. This is the maximal maintainable capital stock: if $k_t > \bar{k}$, then $k_{t+1} \le (1-\delta)k_t + f(k_t) < k_t$. We can therefore limit attention to the compact interval $k \in [0, \bar{k}]$.

With this, the assumptions from the previous lecture hold, and we know that the value function exists, and is unique, continuous, strictly increasing, and strictly concave. However, we don't know much about the policy function except that the sequence it generates, i.e. $(k_0, g(k_0), g(g(k_0)), \dots)$, is optimal from k_0. In the following, we shall see that without solving the model at all, and with only the modest assumptions we made on f and u we can prove that the model has a fixed point and that it is stable.

First, let us redefine the production function so that we are using the same notation as Stokey-Lucas. We define $\tilde{f}(k) = (1-\delta)k + f(k)$ and from here on we drop the tildes. Also, we use the resource constraint to substitute for c, and we write the policy rule explicitly $k' = g(k)$ so we have

$$u'[f(k) - g(k)] = \beta v'(g(k)), \tag{6}$$

$$v'(k) = f'(k)u'[f(k) - g(k)]. \tag{7}$$

To start off, let's show that $g(k)$ is an increasing function. We prove this by contradiction. Let $k_1, k_2 \in (0, \bar{k}]$ be such that $k_1 < k_2$ but $g(k_1) \geq g(k_2)$. Since v is strictly concave

$$u'[f(k_1) - g(k_1)] = \beta v'(g(k_1)) \leq \beta v'(g(k_2)) = u'[f(k_2) - g(k_2)],$$

but since u is also concave

$$f(k_1) - g(k_1) \geq f(k_2) - g(k_2) \Rightarrow g(k_2) - g(k_1) \geq f(k_2) - f(k_1) > 0$$

in contradiction.

The model has a trivial fixed point at $k = 0$, simply because zero capital makes zero product. We also already know that this problem has an additional fixed point, but could we have known this from the recursive formalism? The answer is: yes! Here is how: first note that if there is a fixed point $k^* = g(k^*)$ with $k^* \in (0, \bar{k})$ then the equations imply that $f'(k^*) = 1/\beta$. Notice that we are not yet saying that there is a fixed point, just that if there was, it would have to satisfy this equation. Since f' is monotonically decreasing from $f'(0) = +\infty$ to $f(\bar{k}) < 1 < 1/\beta$, it follows that there is a unique solution to $f'(k^*) = 1/\beta$, which is the only candidate for a nontrivial fixed point.

To show that k^* is indeed a fixed point, notice that because v is strictly concave

$$[v'(k) - v'(g(k))][k - g(k)] \leq 0,$$

with equality if and only if $k = g(k)$. Substituting (6) for $v'(g(k))$ and (7) for $v'(k)$, and using the fact that $u' > 0$, we have

$$[f'(k) - 1/\beta][k - g(k)] \leq 0,$$

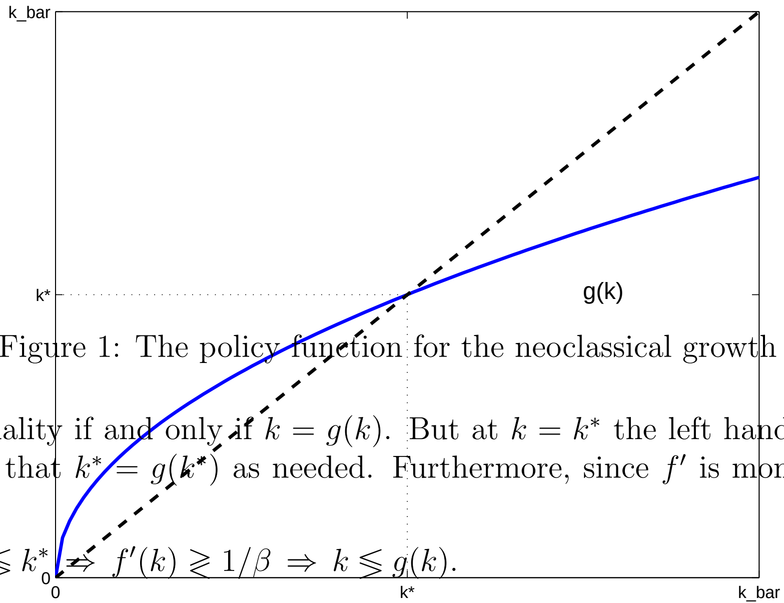

Figure 1: The policy function for the neoclassical growth model.

with equality if and only if $k = g(k)$. But at $k = k^*$ the left hand vanishes, so it must be that $k^* = g(k^*)$ as needed. Furthermore, since f' is monotonic we have that

$$k \lesseqgtr k^* \Rightarrow f'(k) \gtreqless 1/\beta \Rightarrow k \lesseqgtr g(k).$$

Thus, we know that the function g must be as shown in figure 1, which proves that there is global convergence to k^*, as we found in the first lecture.

3.2 Another example

The next example demonstrates that the nice properties of the neoclassical growth model are not general properties of dynamic planning problems. Consider an economy with two goods: a consumption good and a capital good. Each household is endowed with one unit of labor to be split between time spent manufacturing the two goods. The consumption good is manufactured using capital and labor, and the production function is $c = nf(k/n)$ with f satisfying the same conditions as before. The capital good is produces with labor only $k' = 1 - n$.

The functional equation is

$$v(k) = \max_{k' \in [0,1]} \left\{ u\left[(1-k')f\left(\frac{k}{1-k'}\right)\right] + \beta v(k') \right\}.$$

The FOC and envelope condition:

$$u'\left[(1-k')f\Big(\frac{k}{1-k'}\Big)\right]\left[f\Big(\frac{k}{1-k'}\Big)-\frac{k}{1-k'}f'\Big(\frac{k}{1-k'}\Big)\right]=\beta v'(k'),$$
$$v'(k)=u'\left[(1-k')f\Big(\frac{k}{1-k'}\Big)\right]f'\Big(\frac{k}{1-k'}\Big).$$

We search for a steady state $k^*=1-n^*$. Combining the above equations:

$$\Big(\beta+\frac{k^*}{1-k^*}\Big)f'\Big(\frac{k^*}{1-k^*}\Big)-f\Big(\frac{k^*}{1-k^*}\Big)=0.$$

By the assumptions on f, the equation $(\beta+z)f'(z)-f(z)=0$ has a unique solution $z^*>0$. To see this note that the left-hand-side is monotonically decreasing (check that the derivative is negative), that it is positive as $z\to 0$ and negative as $z\to+\infty$, so there must be a unique k^*/n^* that solves the equation. The exact same argument we used before shows that the solution is indeed a steady state. However, unlike the neoclassical growth model, we don't know if the policy function $k'=g(k)$ is monotonous. Clearly, when $k=0$ there is no point in investing any time in producing consumption goods, so $g(0)=1$, therefore the policy function must be decreasing at least near $k=0$. Whether or not the function is decreasing throughout the interval actually depends on the particulars of u and f.