Lecture 2
Dynamic Planning without Uncertainty

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October 25, 2013

1 Formulating the Problem

The general form of the problem we are interested in is

$$\sup_{\{x_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \quad \text{,s.t.} \quad \forall t \in \mathbb{N},\, x_{t+1} \in \Gamma(x_t), \qquad x_0 \in X \quad \text{given.} \tag{1}$$

Here, x_t can be either a scalar or a (more frequently) a vector. Notice that Γ is a mapping from a point x_t at time t to a *set* of feasible choices for the next period. For example, the neoclassical growth model (without labor) can be written in this form by defining: $x_{t+1} = (c_t, k_{t+1})$, $F(x_t, x_{t+1}) = u(c_t)$,

$$\Gamma(c_{t-1}, k_t) = \{(c_t, k_{t+1}) \in \mathbb{R}_+^2 \mid c_t + k_{t+1} \leq (1-\delta)k_t + f(k_t)\},$$

and $X = \mathbb{R}_+^2$.

Assuming that problem (1) is well defined, we can define the value function

$$v(x_0) = \sup_{\{x_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \quad \text{,s.t.} \quad \forall t \in \mathbb{N},\, x_{t+1} \in \Gamma(x_t),$$

for all $x_0 \in X$, and, as before, write the functional equation

$$v(x) = \sup_{y \in \Gamma(x)} \big[F(x,y) + \beta v(y)\big], \quad \forall x \in X. \tag{2}$$

In this lecture we will discuss the relationship between solutions to (1) and to (2). Ideally, we would like to be able to say that if we find a function $v(\bullet)$ that solves (2), then for any $x_0 \in X$, $v(x_0)$ is the value of the maximum of the problem

(1) with that x_0, and that the sequence $\{x_t\}$ attains the maximum in (1) if and only if for all t,

$$v(x_t) = F(x_t, x_{t+1}) + \beta v(x_{t+1}).$$

This turns out to be true under some assumptions that are not too restrictive.

2 Definitions and Assumptions

The components of the problem is a set X, which is the set all possible values for the state variable x. You can think of it as some subset of $\mathbb{R}^n$, but in many applications it is far more general. Next, we have a mapping $\Gamma : X \to 2^X$, assigning for each point in X a subset in X of feasible options for the next period. Let A be the graph of Γ:

$$A = \{(x, y) \in X \times X \mid y \in \Gamma(x)\}.$$

We also have a one-period return function $F : A \to \mathbb{R}$, and a discount factor $\beta \in (0, 1)$.

Call any sequence $\{x_t\}_{t=0}^{\infty}$ a *plan*. Given $x_0 \in X$, define the set of feasible plans from x_0 as

$$\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty} \mid \forall t,\, x_{t+1} \in \Gamma(x_t)\}.$$

Assumption 1. $\Gamma(x)$ is nonempty for all $x \in X$.

Assumption 2. For all $x_0 \in X$ and $\underline{x} \in \Pi(x_0)$, the limit

$$u(\underline{x}) = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})$$

exists in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

One simple way to ensure that assumption (2) holds is for F to be bounded. Then, since $\beta \in (0, 1)$ the limit always exists.

Assumption 1 assures us that $\Pi(x_0)$ is nonempty, and assumption 2 assures that for any $\underline{x} \in \Pi(x_0)$, the function $u : \Pi(x_0) \to \overline{\mathbb{R}}$ is well defined. The next lemma then follows:

Lemma 1. *Let X, Γ, F, β satisfy assumptions 1-2. Then for any $x_0 \in X$ and $\underline{x} \in \Pi(x_0)$,*

$$u(\underline{x}) = F(x_0, x_1) + \beta u(\underline{x}'),$$

where $\underline{x}' = (x_1, x_2, \ldots)$.

Proof. By assumption 2, the infinite series converges, therefore any tail of the series also converges. In particular,

$$u(\underline{x}) = \lim_{T\to+\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) = F(x_0, x_1) + \lim_{T\to+\infty} \sum_{t=1}^{T} \beta^t F(x_t, x_{t+1}) =$$

$$= F(x_0, x_1) + \beta \lim_{T\to+\infty} \sum_{t=0}^{T-1} \beta^t F(x_{t+1}, x_{t+2}) = F(x_0, x_1) + \beta u(\underline{x}').$$

□

3 Equivalence of (1) and (2)

Theorem 2. *Let X, Γ, F, β satisfy assumptions 1-2. Then the function v^*:*

$$v^*(x_0) = \sup_{\underline{x}\in\Pi(x_0)} u(\underline{x}) \tag{3}$$

satisfies the functional equation (2).

Proof. First assume that $v^*(x_0)$ is finite. In order to prove that the functional equation (2) holds, we must prove that

$$\forall x_1 \in \Gamma(x_0), \quad v^*(x_0) \geq F(x_0, x_1) + \beta v^*(x_1), \tag{4}$$

and that for all $\epsilon > 0$,

$$\exists x_1 \in \Gamma(x_0), \quad v^*(x_0) - \epsilon < F(x_0, x_1) + \beta v^*(x_1). \tag{5}$$

To prove (4), let $x_1 \in \Gamma(x_0)$ and let $\delta > 0$. By the definition of the supremum (3), there exists some $\underline{x}' = (x_1, x_2, \dots) \in \Pi(x_1)$ such that

$$u(\underline{x}') > v^*(x_1) - \delta.$$

However, since $x_1 \in \Gamma(x_0)$ and $\underline{x}' \in \Pi(x_1)$, it follows that $\underline{x} = (x_0, x_1, x_2, \dots) \in \Pi(x_0)$, therefore $v^*(x_0) \geq u(\underline{x})$. Using Lemma 1,

$$v^*(x_0) \geq u(\underline{x}) = F(x_0, x_1) + \beta u(\underline{x}') > F(x_0, x_1) + \beta v^*(x_1) - \beta\delta.$$

Since the above is true for any $\delta > 0$, equation (4) holds.

To prove (5), let $\epsilon > 0$. By the definition of the supremum (3), there exists some $\underline{x} = (x_0, x_1, x_2, \dots) \in \Pi(x_0)$ such that $v^*(x_0) - \epsilon < u(\underline{x})$. Therefore

$$v^*(x_0) - \epsilon < u(\underline{x}) = F(x_0, x_1) + \beta u(\underline{x}') \leq F(x_0, x_1) + \beta v^*(x_1),$$

where in the last step we used the definition (3) to assert that $v^*(x_1) \geq u(\underline{x}')$.

The cases where $v^*(x_0)$ are plus or minus infinity are left as an exercise. □

Theorem 2 says that if we solved the sequential problem (1) and constructed the value function, it will solve the functional equation (2). The next theorem says that the converse is also true, with one caveat:

Theorem 3. *Let X, Γ, F, β satisfy assumptions 1-2, and let v be a solution to the functional equation (2). If for all $x_0 \in X$ and $(x_0, x_1, \dots) \in \Pi(x_0)$, v also satisfies*

$$\lim_{T \to \infty} \beta^T v(x_T) = 0, \tag{6}$$

then $v = v^*$.

Proof. Suppose that v solves (2) and that the condition (6) holds. Fix $x_0 \in X$ and assume $v(x_0)$ is finite. Consider any $\underline{x} = (x_0, x_1, \dots) \in \Pi(x_0)$. Since v solves (2),

$$\begin{aligned} v(x_0) &\geq F(x_0, x_1) + \beta v(x_1) \\ &\geq F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 v(x_2) \\ &\vdots \\ &\geq \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) + \beta^T v(x_T) \xrightarrow[T \to +\infty]{} u(\underline{x}). \end{aligned}$$

In taking the limit in the last step we used assumption 2 and condition (6).

Next, let $\epsilon > 0$. Since v solves (2), we can find $x_1 \in \Gamma(x_0)$, $x_2 \in \Gamma(x_1)$, $x_3 \in \Gamma(x_2)$, and so on, such that for all $t = 0, 1, \dots$

$$v(x_t) < F(x_t, x_{t+1}) + \beta v(x_{t+1}) + (1 - \beta)\epsilon.$$

Therefore,

$$\begin{aligned} v(x_0) &< F(x_0, x_1) + \beta v(x_1) + (1 - \beta)\epsilon \\ &< F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 v(x_2) + (1 - \beta)\epsilon + \beta(1 - \beta)\epsilon \\ &\vdots \\ &< \sum_{t=0}^{T-1} \beta^t F(x_t, x_{t+1}) + \beta^T v(x_T) + (1 - \beta)\epsilon \sum_{t=0}^{T-1} \beta^t \xrightarrow[T \to +\infty]{} u(\underline{x}) + \epsilon. \end{aligned}$$

In conclusion, we have that $v(x_0) > u(\underline{x})$ for all $\underline{x} \in \Pi(x_0)$, and that for any $\epsilon > 0$, there exists some $\underline{x} \in \Pi(x_0)$ such that $v(x_0) < u(\underline{x}) + \epsilon$, therefore $v(x_0) = \sup_{\underline{x} \in \Pi(x_0)} u(\underline{x})$. Since the supremum is unique, we have $v = v^*$.

Again, the case where $v(x_0)$ is not finite is left as an exercise. □

The two theorems we have proven so far tell us that we can find the value function be solving the optimization problem in the sequential formulation (1) or by solving the functional equation (2), and that the the two are equivalent as long as the condition (6) holds. However, in most applications the object of interest is not the value function but the optimizing sequence. The next two theorems relate these optimal sequences in the two approaches.

We say that a sequence $\underline{x} \in \Pi(x_0)$ is *optimal plan from* x_0 if it attains the supremum in (1): $u(\underline{x}) = v^*(\underline{x})$.

Theorem 4. *Let* X, Γ, F *and* β *satisfy assumptions 1-2. Let* $\underline{x}^* \in \Pi(x_0)$ *be an optimal plan from* x_0*, then*

$$\forall t \in \mathbb{N}, \quad v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*). \tag{7}$$

Proof. Let $\underline{x}^*$ be an optimal plan from x_0. Consider a sequence $\underline{\tilde{x}} \in \Pi(x_0)$ that starts with the same two values as $\underline{x}^*$, but is otherwise arbitrary: $\underline{\tilde{x}} = (x_0, x_1^*, \tilde{x}_2, \tilde{x}_3 \dots)$. Since $\underline{x}^*$ is optimal and by Lemma 1

$$\begin{aligned} u(\underline{x}^*) &\geq u(\underline{\tilde{x}}) \\ F(x_0, x_1^*) + \beta u(\underline{x}'^*) &\geq F(x_0, x_1^*) + \beta u(\underline{\tilde{x}}') \\ u(\underline{x}'^*) &\geq u(\underline{\tilde{x}}'). \end{aligned}$$

Since $\underline{\tilde{x}}'$ can be any sequence in $\Pi(x_1^*)$, we have that $\underline{x}'^*$ is the optimal plan from x_1, therefore, $u(\underline{x}^{*\prime}) = v(x_1^*)$. Thus,

$$v^*(x_0) = u(\underline{x}^*) = F(x_0, x_1^*) + \beta u(\underline{x}'^*) = F(x_0, x_1^*) + \beta v(x_1^*),$$

which is (7) with $t = 0$. Continuing by induction establishes (7) for all t. □

Theorem 5. *Let* X, Γ, F *and* β *satisfy assumptions 1-2. Let* $\underline{x}^* \in \Pi(x_0)$ *be a feasible plan from* x_0 *that satisfies* (7)*, and also*

$$\limsup_{t \to \infty} \beta^t v^*(x_t^*) \leq 0. \tag{8}$$

Then $\underline{x}^*$ *is an optimal plan from* x_0*.*

Proof. Let $\underline{x}^* \in \Pi(x_0)$ satisfy (7), then

$$\begin{aligned} v^*(x_0) &= F(x_0, x_1^*) + \beta v^*(x_1^*) \\ &= F(x_0, x_1^*) + \beta F(x_1^*, x_2^*) + \beta^2 v^*(x_2^*) \\ &\vdots \\ &= \sum_{t=0}^{T-1} \beta^t F(x_t^*, x_{t+1}^*) + \beta^T v^*(x_T^*). \end{aligned}$$

By taking $T \to +\infty$, we find $v^*(x_0) \leq u(\underline{x}^*)$. Of course, by definition of v^*, $v^*(x_0) \geq u(\underline{x}^*)$, therefore $v^*(x_0) = u(\underline{x}^*)$ and $\underline{x}^*$ is an optimal plan. □

Theorems 4 and 5 tell us that if we know the value function, we can verify that a sequence is an optimal plan by checking that every two sequential elements satisfy (7). Furthermore, if we have a sequence that satisfies (7) and also (8), then it must be an optimal plan.

4 Existence, Uniqueness and so on

We have so far talked about the relationships between the value functions and the optimal plans of problems (1) and (2), but we have not made any claims about the existence or the uniqueness of solutions to either problems. There are various assumptions that one can add that will guarantee existence of a solution. We will not prove this results here,[1] but simply state the assumptions.

Assumption 3. X is a convex subset of $\mathbb{R}^l$, and for all $x \in X$, $\Gamma(x)$ is nonempty, compact and continuous.[2]

Assumption 4. The function $F : A \to \mathbb{R}$ is bounded and continuous.

Assumptions 3 and 4 are clearly stronger than assumptions 1-2, and so the theorems we have proved so far hold. Under the new assumptions one can show that the functional equation (2) has unique solution, and that it satisfies (6), so that it is also the value function of (1).

[1] The proofs and further discussion can be found in Chapter 3 of Stokey-Lucas.

[2] A nonempty compact valued correspondence Γ is said to be upper-hemicontinuous if the graph $A = \{(x, y) \,|\, y \in \Gamma(x)\}$ is closed. Is is lower-hemicontinuous at x if for any open set V such that $V \cap \Gamma(x) \neq \emptyset$, there exists a neighborhood U such that for all $x' \in U$, $\Gamma(x') \cap V \neq \emptyset$. Γ is continuous if it is both upper and lower hemicontinuous.

5 Example: Monopolistic Competition with Adjustment Costs

Consider a monopolistic firm that faces an inverse demand schedule of $p = a_0 - a_1 q$. Production is costless, but there is cost to adjust production given by $d \cdot (q_t - q_{t-1})^2/2$. The firm's period profits is thus $\pi_t = (a_0 - a_1 q_t)q_t - d \cdot (q_t - q_{t-1})^2/2$. The wishes to maximize the discounted sum of its profits:

$$U = \sum_{t=0}^{\infty} \beta^t \pi_t = \sum_{t=0}^{\infty} \beta^t \left[(a_0 - a_1 q_t) q_t - \frac{d}{2} \cdot (q_t - q_{t-1})^2 \right].$$

The Euler equation is

$$\beta d q_{t+2} - (2a_1 + d + \beta d) q_{t+1} + d q_t = -a_0.$$

This is a linear second order difference equation. To find the general solution we solve the homogenous equation: $\gamma_2 q_{t+2} - \gamma_1 q_{t+1} + \gamma_0 q_t = 0$, by guessing $q(t) = \lambda^t$:

$$\gamma_2 \lambda^{t+2} - \gamma_1 \lambda^{t+1} + \gamma_0 \lambda^t = 0 \Rightarrow \gamma_2 \lambda^2 - \gamma_1 \lambda + \gamma_0 = 0 \Rightarrow \lambda_\pm = \frac{\gamma_1 \pm \sqrt{\gamma_1^2 - 4\gamma_0\gamma_2}}{2\gamma_1}.$$

We find the special solution by guessing $q_t = q^*$ and finding

$$\beta d q^* - (2a_1 + d + \beta d) q^* + d q^* = -a_0 \Rightarrow q^* = \frac{a_0}{2a_1}.$$

The general solution is therefore

$$q_t = \frac{a_0}{2a_1} + C_+ \lambda_+^t + C_- \lambda_-^t.$$

Recall that the product of two roots of a quadratic equation is $\gamma_0/\gamma_2 = 1/\beta > 1$, so both roots are of the same sign. Furthermore, the sum is $\gamma_1/\gamma_0 = 1 + 1/\beta + 2a_1/(\beta d) > 2$, so both roots are positive and at least one of them is larger than one. It is in fact not difficult to verify that $\lambda_+ > 1$ and $0 < \lambda_- < 1$. Since the solution must not diverge, we must have $C_+ = 0$. Finally substituting $t = 0$ in the general solution, we find $C_- = q_0 - a_0/(2a_1)$, therefore the solution is

$$q_t = \frac{a_0}{2a_1} + \left(q_0 - \frac{a_0}{2a_1} \right) \lambda_-^t.$$

Notice that q_t approaches $q^* = a_0/(2a_1)$, which is the optimal choice for a monopolist. The solution we found describes how quickly the firm approaches q^* when faced with adjustment costs.

The functional equation for this problem is

$$v(q) = \max_{q'} \left[(a_0 - a_1 q')q' - \frac{d}{2}(q' - q)^2 + \beta v(q') \right].$$

The FOC is

$$a_0 - 2a_1 q' - d(q' - q) + \beta v'(q') = 0,$$

and the envelope equation is

$$v'(q) = d(q' - q).$$

The form of these equations suggests that a quadratic value function is appropriate. Furthermore, v should be maximized when $q = q^*$, so we guess $v(q) = w_2 q(q - 2q^*) + w_0$. Substituting this into the envelope condition gives q' as a function of q and w_2. Using this in the FOC gives a linear equation for q. Since this must hold for all q, comparing the coefficients of q to zero determines w_2.