Lecture 1

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1 The neoclassical growth model - Centralized

An economy is populated by many identical household that derive utility from consumption of a single good:

$$U(c) = \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where $\beta \in (0,1)$, and $u : \mathbb{R}^+ \to \mathbb{R}$ satisfies

$$u'(\bullet) > 0, \qquad u''(\bullet) \leq 0, \qquad u'(0) = +\infty, \qquad u'(+\infty) = 0. \tag{2}$$

The consumption good is produced using capital and labor. The production function is $y_t = F(k_t, n_t)$ and the output can be used for either consumption or as new capital. The production function is increasing in both arguments, concave, and homogeneous of degree one (i.e. for all λ, $F(\lambda k, \lambda n) = \lambda F(k,n)$. Also, $F(0,n) = F(k,0) = 0$.

The initial capital stock is k_0 and capital depreciates at a rate $\delta \in [0,1]$. Households are endowed with a unit of leisure, which can be allocated to labor. The planners problem is to choose sequences of consumption $\{c_t\}_{t=0}^{\infty}$, capital $\{k_{t+1}\}_{t=0}^{\infty}$ and labor $\{n_t\}_{t=0}^{\infty}$ to maximize the households utility subject to the resource constraints, the nonnegativity constraints, and the initial conditions. Formally:

$$\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \quad \forall t \in \mathbb{N}, c_t \geq 0,\ k_t \geq 0, 0 \leq n_t \leq 1, \tag{3}$$
$$k_{t+1} = (1-\delta)k_t - c_t + F(k_t, n_t)$$

1.1 The Euler equation

For the time being we are going to take a somewhat naive approach and treat this as a standard many-variable optimization problem and ignore the fact that there are in fact an infinite amount of variables. Since households have no disutility from labor, it is clearly optimal to set $n_t = 1$. Denote $f(k) = F(k, 1)$ If we further ignore the nonnegativity constraints, we can rewrite the problem in lagrangian form

$$L = \sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t) + \lambda_t \Big[(1-\delta)k_t - c_t + f(k_t) - k_{t+1} \Big] \Big\}. \tag{4}$$

What we have done here is added an infinite series of lagrange multiplier, one for each resource constraint. The lagrange multiplier for the resource constraint at time t is $\beta^t \lambda_t$. By taking first order conditions we get the following infinite set of equations

$$u'(c_t) - \lambda_t = 0, \tag{5a}$$

$$-\lambda_t + \beta\lambda_{t+1}[(1-\delta) + f'(k_{t+1})] = 0, \tag{5b}$$

$$\boxed{k_{t+1} = (1-\delta)k_t + f(k_t) - c_t.} \tag{5c}$$

Combining equations (5a) and (5b) allows us to get rid of the multipliers:

$$\boxed{u'(c_t) = \beta u'(c_{t+1})[(1-\delta) + f'(k_{t+1})].} \tag{6}$$

The above is known as the intertemporal Euler equation. To understand this equation consider the following change to the optimal path: at time t reduce consumption from c_t to $c_t - \epsilon$, and at time $t+1$ increase consumption to get back on the optimal path. The left-hand-side of (6) is the marginal loss of utility at time t. The extra saving means that at time $t+1$ production would increase by $\epsilon F'(k_{t+1})$, and on top of that we will have $(1-\delta)\epsilon$ extra units of goods to consume in order to return to the optimal path. Thus, overall the marginal increase to consumption at $t+1$ is $(1-\delta)+f'(k_{t+1})$. The right-hand-side of (6) is the marginal increase of utility flow at $t+1$ from this extra consumption discounted (since households discount delayed consumption by a factor of β for each period).

What is striking about the Euler equation is that it implies that this single type of deviation, reducing consumption at t and increasing it on $t+1$, is the only one we need to consider. This is in fact a special case of the so called one-shot deviation principle.[1]

[1] Blackwell, David. Discounted Dynamic Programming. The Annals of Mathematical Statistics 36, no. 1 (Feb 1, 1965): 226-235.

1.2 Steady State

The system defined by the Euler equation (6) and the resource constraint (5c) admits one interesting solution. It is one where k_t and c_t are constant for all t. To find it, simply replace $c_t = c_{t+1} = c^*$ and $k_{t+1} = k^*$ into the Euler equation. The result is

$$f'(k^*) = \delta + \frac{1-\beta}{\beta}. \tag{7}$$

The resource constraint can then be used to find c^*. You might recall the Solow model from your undergraduate education. There the golden rule for the capital stock was $f'(k^*) = \delta$, i.e. we don't want to grow the capital stock past the point where the additional output is less than what would be needed to replace the depreciating capital. In the steady-state equation (7) the right-hand-side is larger than δ, so k^* is lower. The reason is that unlike the Solow mode, our agents are impatient: an extra unit of consumption today is worth $u'(c^*)$, while an extra unit of consumption starting next period and to the end of time is worth $u'(c^*) \cdot \beta/(1-\beta)$.

We shall see below that for any initial value k_0, the optimal growth path converges to the stationary point (c^*, k^*).

1.3 Solving the Euler equation

The Euler equation (6) and the resource constraint (5c) together define a mapping from (c_t, k_t) to (c_{t+1}, k_{t+1}). This is how it works: the resource constraint gives us k_{t+1} explicitly in terms of (c_t, k_t), and using that we solve the Euler equation to find c_{t+1}. Therefore, since k_0 is given, any choice of c_0 defines a path of capital and consumption for all time. So only using (5c) and (6), it seems that we can construct infinitely many "optimal" paths. However, if you try and follow this recipe (and you should), you'll find that for most values of c_0 the path you end up with cannot possibly be optimal for one of two reasons.

To better illustrate this point, it is useful to draw a phase diagram (see figure 1). We plot k_t on the horizontal axis and c_t on the vertical, and from each point in the (c_t, k_t) plane we draw a vector to the point (c_{t+1}, k_{t+1}) that the above mapping defines. Starting from any value of k_0 we see that if we choose c_0 to be too high, we end up crashing on the $k = 0$ line. If we choose c_0 to be too low, we converge toward the $c = 0$ line, but whether or not that happens in finite time depends on the details of the model. There is one unique value of c_0 that puts the system on a path toward the stationary point that we found above.

I'm not going to go into how one draws a phase diagram. The curious reader

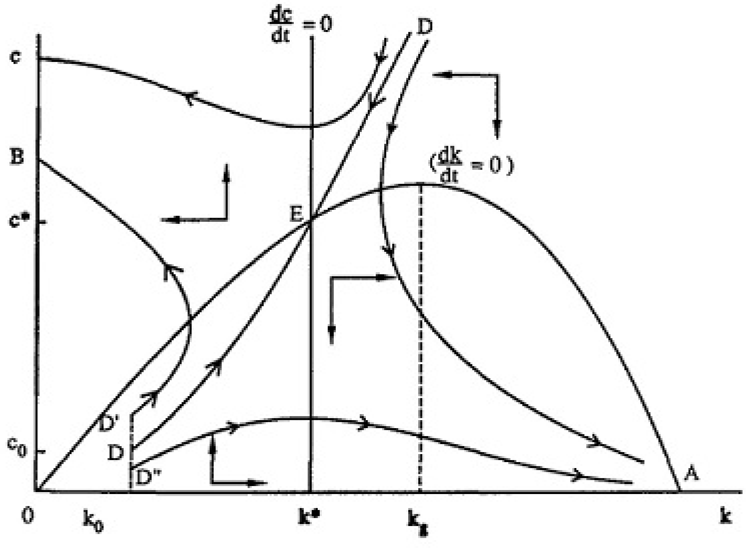

Figure 1: A phase diagram for the neoclassical growth model. The figure is from Blanchard and Fischer chapter 2.

can find that in Blanchard and Fischer chapter 2.[2] Analyzing phase diagrams is a technique which is not used very frequently in modern macroeconomics. It is a powerful technique to study the global behaviour of systems, especially if the system has multiple equilibria.[3] Instead, let's pay a little more attention to the result.

Since $u'(0) = +\infty$ and since if k ever hits zero it remains zero forever, the paths that violate one of the nonnegativity constraints are clearly not optimal. The paths that tend toward $c = 0$ but in infinite time require more considerations. The paths in question all tend toward the point where $f(k) = \delta k$, i.e. the maximal level of the capital stock (where the entire production just replaces the depreciating capital and no consumption takes place). This is a peculiarity that arises due to the infinite nature of the planning problem. At each point in time the agent can be convinced to forgo a little bit more consumption in exchange for a lot of consumption at the end of time. Of course, the end of time never arrives.

To rule out these paths we need an additional condition that is called the transversality condition. If the agents in the problem only lived a finite amount of periods T, it would clearly be optimal to consume the entire capital stock at the last period and fix $k_{T+1} = 0$. In the infinite horizon problem we don't want to impose that the capital stock must go to zero, because there are always infinitely many period ahead. Instead, we require that the present-discounted value of the capital stock vanishes at infinity. Recall that the lagrange multiplier on the time t constraint, $\beta^t \lambda_t$, is the shadow price of capital at time $t+1$, so the requirement is

$$0 = \lim_{t \to \infty} \beta^t \lambda_t k_{t+1} = \lim_{t \to \infty} \beta^t u'(c_t) k_{t+1}. \tag{8}$$

We will talk more about this later, but for now I'll just state that the unique solution to planner's problem (3) with the additional constraint (8) is the path that converges toward the stationary point.

[2] Blanchard, Olivier, and Stanley Fischer. *Lectures on Macroeconomics.* Cambridge, Mass.: MIT Press, 1989.

[3] For a recent example, see Kaplan, Greg, and Guido Menzio. *Shopping Externalities and Self-Fulfilling Unemployment Fluctuations.* Working Paper. NBER, February 2013. http://www.nber.org/papers/w18777.

2 The neoclassical growth model - Decentralized

We now turn to the decentralized version. As before, there is a large number of identical households that derive utility from consumption of a single good:

$$U(c) = \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where u satisfies the same conditions as before (2). Households own the capital stock and rent it out to firms for an interest rate r_t, and also provide labor for a wage w_t. Furthermore, the household also own the firms, and any profits are assumed to be distributed at each period. The household's budget constraint is therefore

$$c_t + k_{t+1} = (1 - \delta + r_t)k_t + w_t n_t + \pi_t.$$

There is a large number of identical firms. Firms choose labor and capital to maximize profits:

$$\pi_t = \max_{k_t, n_t} \Big[F(k_t, n_t) - r_t k_t - w_t n_t \Big]$$

The firm's FOCs are

$$F_k(k_t, n_t) = r_t$$
$$F_n(k_t, n_t) = w_t.$$

For the households, we write the lagrangian

$$L = \sum_{t=0}^{\infty} \beta^t \Big\{ u(c_t) + \lambda_t \big[(1 - \delta + r_t)k_t + w_t n_t + \pi_t - c_t - k_{t+1} \big] \Big\}$$

The households only choose c_t and k_t (subject to the budget constraint) – they are price takers. They treat w_t, r_t and π_t as given (they are also too small to influence the firm's decision). They're FOCs are therefore

$$u'(c_t) - \lambda_t = 0,$$
$$-\lambda_t + \beta \lambda_{t+1} \big[1 - \delta + r_{t+1} \big],$$
$$n_t = 1.$$

(the last one is just a result of households having no disutility from labor as before). Here too we add the transversality condition

$$\lim_{t \to \infty} \beta^t u'(c_t) k_t = 0.$$

Definition 1. *A **competitive equilibrium** of the decentralized problem are series* $\{c_t, k_{t+1}, n_t, \pi_t, r_t, w_t\}_{t=0}^{\infty}$ *such that given* k_0*:*

1. *Households maximize utility for the given paths of* r_t, w_t *and* π_t*.*
2. *Firms maximize profits for the given paths of* r_t *and* w_t*.*
3. *Markets for capital and labor clear.*

In other words, a competitive equilibrium is a solution to the following system of equations (not worrying about corner solutions, etc.)

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + r_{t+1}], \tag{9a}$$
$$c_t + k_{t+1} = (1 - \delta + r_t)k_t + w_t n_t, \tag{9b}$$
$$n_t = 1, \tag{9c}$$
$$\lim_{t\to\infty} \beta^t u'(c_t)k_t = 0, \tag{9d}$$
$$F_k(k_t, n_t) = r_t \tag{9e}$$
$$F_n(k_t, n_t) = w_t \tag{9f}$$

The first three equations come from the household's problem, the last two from the firm's problem. The market clearing conditions are implicitly used when we use the same variable k_t (n_t) to denote both the capital (labor) supply and demand. Plugging equation (9e) into (9b) and using $n_{t+1} = 1.$, we get

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + f'(k_{t+1})],$$

which is identical to the central planner's Euler equation (6). Using (9e) and (9f) in (9b), we get

$$c_t + k_{t+1} = (1 - \delta)k_t + F_k(k_t, n_t)k_t + F_n(k_t, n_t)n_t.$$

Since $F(k, n)$ is homogenous of degree 1, by Euler's homogenous function theorem,[4]

$$F_k(k_t, n_t)k_t + F_n(k_t, n_t)n_t = F(k_t, n_t).$$

Therefore we find (using $n_t = 1$)

$$c_t + k_{t+1} = (1 - \delta)k_t + f(k_t),$$

which is identical to the central planner's resource constraint (5c). Finally, the transversality condition (9d) is identical to (8). The conclusion is summarized in the following thoerem.

[4]See the appendix.

Theorem 1. *The first and second welfare theorems:*

1. *If* $\{c_t, k_{t+1}, n_t, \pi_t, r_t, w_t\}_{t=0}^{\infty}$ *is a competitive equilibrium, then* $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$ *is solution to the planner's problem, and therefore it is Pareto efficient.*

2. *For any (Pareto efficient) solution to the planner's problem* $\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}$ *there exist series* $\{w_t, r_t, \pi_t\}_{t=0}^{\infty}$ *such that* $\{c_t, k_{t+1}, n_t, \pi_t, r_t, w_t\}_{t=0}^{\infty}$ *is a competitive equilibrium.*

3 Two equilibrium concepts

There are two ways to think about the decentralized problem: in the first all decisions and trade happens at $t = 0$, and in the later decisions about c_t, n_t and k_{t+1} happen at time t. In the first approach we imagine that each household start out at time 0 with infinitely many pieces of paper each saying "one unit of labor at time t" for $t = 0, 1, \ldots$. Additionally it has a certain quantity of the good (their initial endowment) which it deposits with a Walrasian auctioneer for an equal amount of pieces of paper that say "one unit of the good at time 0" (remember that the good can be either consumed or used as capital). In the auction hall there are also representatives of the firms. The firms are allowed to transform claims to time $t-1$ capital and time t labor into claims to time t goods. All the agents meet at $t = 0$ and trade these claims in the standard Walrasian way (the auctioneer sets prices so that the markets for goods and labor at all time periods clear). For the rest of time, they just deliver goods and labor according to the claims. This is the Arrow-Debreau approach.

A perhaps more natural alternative is to imagine that trade occurs at the beginning of each period. This is the sequential approach. Do the two approached yield the same results? Another way to ask this is: suppose that we allowed the agents from the previous problem to meet again at $t = 1$ and trade, would they make any changes? The answer to this question is not trivial: the agents at time $t = 0$ are maximizing

$$U_0(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t),$$

with a set of constraints and k_0 given. If the agents at time $t = 1$ were maximizing U_0 with c_0 (and thus k_1) given, then the answer would obviously be that they will not change anything. But the agents at $t = 1$ are maximizing something else:

$$U_1(c_1, c_2, \ldots) = \sum_{t=1}^{\infty} \beta^{t-1} u(c_t),$$

with k_1 given. Luckily, as it happens, the above is the same utility function up to a multiplicative constant (β), so the optimization does in fact lead to the same result. To see this, let's define the value function:

$$V(k_0) = \max_{\{c_t\}_{t=0}^{\infty}} U_0(\{c_t\}_{t=0}^{\infty}) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{10}$$

where the maximization is under the constraints $c_t \geq 0$, $k_t \geq 0$, and

$$c_0 + k_1 = (1-\delta)k_0 + f(k_0) \tag{11a}$$
$$c_1 + k_2 = (1-\delta)k_1 + f(k_1) \tag{11b}$$
$$\vdots$$
$$c_t + k_{t+1} = (1-\delta)k_t + f(k_t) \tag{11c}$$
$$\vdots$$

Next, we break the maximization into two parts: first over $\{c_t\}_{t=1}^{\infty}$ and then over c_0:

$$V(k_0) = \max_{c_0}\left\{\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)\right\} = \max_{c_0}\left\{\max_{\{c_t\}_{t=1}^{\infty}} \left[u(c_0) + \sum_{t=1}^{\infty} \beta^t u(c_t)\right]\right\} =$$
$$= \max_{c_0}\left\{u(c_0) + \max_{\{c_t\}_{t=1}^{\infty}} \sum_{s=0}^{\infty} \beta^{s+1} u(c_{s+1})\right\} = \max_{c_0}\left\{u(c_0) + \beta \max_{\{c_t\}_{t=1}^{\infty}} \sum_{s=0}^{\infty} \beta^{s} u(c_{s+1})\right\}.$$

Consider the last term in the bottom line. It asks us to find a series $(c_1, c_2, \dots)$ to maximize a function that is exactly the same as the one we are maximizing in the definition of (10). Since the constraint (11a) is irrelevant for the inner maximization problem, and the rest of the constraints (11b)-(11c)... take the exact same form. Thus,

$$V(k_0) = \max_{c_0}\left[u(c_0) + \beta V(k_1)\right].$$

The more common way of writing this equation is

$$V(k) = \max_{c,k'}\left[u(c) + \beta V(k')\right], \quad \text{s.t.} \quad c + k' = (1-\delta)k + f(k) \tag{12}$$

The variable k is called the *state variable*. The set of state variables of a problem are essentially all the information you need at time t to solve optimization problem. c is the *choice variable*, and the constraint is sometimes called the *evolution equation*, because it tells us how the state variables evolve over time depending on the choice variable. For any value of the state variable k the agent maximizes the sum of her present period utility and the discounted *continuation value*. For any k, the maximization problem defines an optimal choice $c^*(k)$, which is called the *policy function*.

The recursive formulation of the problem (12) is a functional equation. If we knew the function $V(\bullet)$, then we could find the optimal choice of c, but we normally don't. Solving this problem is thus not at all straightforward. We can however find two relationships: since c is maximizing $u(c) + V(k')$ subject to the constraint, the FOCs tell us (assuming an internal solution)

$$\boxed{u'(c^*(k)) = \beta V'(k').} \tag{13}$$

The second relationship comes from the envelope theorem.[5] Since the lagrangian of this problem is

$$L = u(c) + \beta V(k') - \lambda(c + k' - (1-\delta)k - f(k)),$$

then the envelope theorem is

$$V'(k) = \lambda(1 - \delta + f'(k)).$$

Using the FOCs for c to eliminate λ, we get the second relationship:

$$\boxed{V'(k) = [1 - \delta + f'(k)]u'(c^*(k)).} \tag{14}$$

Equation (13), the FOC equation, states that the marginal utility of consuming current output is equal to the the marginal utility of allocating it to future production. Equation (14), the envelope condition, states that the marginal value of current capital is the marginal utility that can be obtained by producing more and consuming it today.

In the next lecture we will discuss conditions under which the solutions to the sequential problem are the same as the solution to the recursive problem.

A Euler's Homogenous Function Theorem

We say that a function $f : \mathbb{R}^k \to \mathbb{R}$ is homogenous of degree n if for all $x \in \mathbb{R}^k$ and $\lambda > 0$

$$f(\lambda x) = \lambda^n f(x).$$

[5]If you don't know the envelope theorem, take a look at the appendix to these notes. It is going to be very used extensively in the recursive approach.

Euler's Homogenous Function Theorem. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a differentiable function, then f is homogenous of degree n if and only if for all $x \in \mathbb{R}^k$*

$$\sum_{i=1}^{k} x_k \cdot \frac{\partial}{\partial x_k} f(x) = nf(x).$$

Proof. ($\Rightarrow$) Fix x and define $g(\lambda) = f(\lambda x) - \lambda^n f(x)$. Note that g is differentiable for all $\lambda > 0$, and that

$$g'(\lambda) = \sum_{i=1}^{k} x_k \cdot \frac{\partial}{\partial x_k} f(\lambda x) - n\lambda^{n-1} f(x).$$

If f is homogenous of degree n, then $g(\lambda) = 0$ by definition, and since g is constant, $g'(\lambda) = 0$. However,

$$0 = g'(\lambda) = \sum_{i=1}^{k} x_k \cdot \frac{\partial}{\partial x_k} f(\lambda x) - n\lambda^{n-1} f(x).$$

Evaluating the above at $\lambda = 1$ proves the "only if" part of the statement.

For the ($\Leftarrow$) part, let f be such that for all $x \in \mathbb{R}^k$,

$$\sum_{i=1}^{k} x_k \cdot \frac{\partial}{\partial x_k} f(x) = nf(x).$$

Define $g(\lambda) = f(\lambda x) - \lambda^n f(x)$ as before, and note that $g(1) = 0$. Furthermore,

$$\begin{aligned} g'(\lambda) &= \sum_{i=1}^{k} x_k \cdot \frac{\partial}{\partial x_k} f(\lambda x) - n\lambda^{n-1} f(x) = \\ &= \lambda^{-1} \left(\sum_{i=1}^{k} \lambda x_k \cdot \frac{\partial}{\partial x_k} f(\lambda x) \right) - n\lambda^{n-1} f(x) = \\ &\qquad = \lambda^{-1} n f(\lambda x) - n\lambda^{n-1} f(x) = \lambda^{-1} n g(\lambda). \end{aligned}$$

We arrived at a first-order linear ordinary differential equation $g'(\lambda) = \lambda^{-1} n g(\lambda)$. This is solved by $g(\lambda) = C\lambda^n$, and the initial condition $g(1) = 0$ determines the constant of integration to be $C = 0$. Thus, we have $g(\lambda) = 0$, which proves the statement. □

B The Envelope Theorem

We present here two simple versions of the envelope theorem, without and with constraints.

Envelope Theorem 1. *Consider some differentiable function* $f(x,t)$ *whose domain is a simply-connected subset of* $\mathbb{R}^k \times \mathbb{R}$. *Define* $v(t) = \max_x f(x,t)$, *and denote the maximizer by* $x^*(t)$. *Assume that* $x^*(t)$ *is unique and interior for every* t. *Then*

$$\frac{d}{dt}v(t) = \frac{\partial}{\partial t} f(x,t)\Big|_{x=x^*(t)}.$$

Proof. Since we are assuming that f is differentiable and that the maximizer is always interior, we have the FOCs:

$$0 = \frac{\partial}{\partial x_i} f(x,t)\Big|_{x=x^*(t)}.$$

Also, by definition $v(t) = f(x^*(t),t)$, so

$$v'(t) = \left[\frac{\partial}{\partial t} f(x,t)\Big|_{x=x^*(t)} + \sum_{i=1}^{k} \frac{\partial}{\partial x_i} f(x,t)\Big|_{x=x^*(t)} \frac{d}{dt}x_i^*(t)\right] = \frac{\partial}{\partial t} f(x,t)\Big|_{x=x^*(t)},$$

where in the last step we have substituted in the FOCs. □

Envelope Theorem 2. *Consider the constrained maximization problem*

$$v(t) = \max_x f(x,t) \quad s.t. \quad g_n(x,t) = 0,\ n = 1, 2, \ldots$$

The function f *is as before and assume that* g *is also differentiable and that* $x^*(t)$ *is unique and interior for every* t. *Define the lagrangian of the problem*

$$L = f(x,t) - \sum_n \lambda_n g_n(x,t)$$

Then

$$\frac{d}{dt}v(t) = \frac{\partial}{\partial t} L(x,\lambda,t)\Big|_{x=x^*(t),\lambda=\lambda^*(t)}.$$

The proof is similar.