# Macroeconomics A for MA – 57989
# Final Exam - Moed A

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## A. Balanced growth with consumption habits

Consider the problem of choosing a consumption sequence $\{c_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t [\log c_t + \gamma \log c_{t-1}], \quad \beta \in (0,1), \gamma > 0,$$

subject to $c_t + k_{t+1} \leq A k_t^{\alpha}$, $A > 0, \alpha \in (0,1)$, and $k_0$ and $c_{-1}$ given.

1. Write down a recursive formulation of this problem.
2. Write down the first-order conditions and envelope equations associated with the problem.
3. Show that there exist three constants $E, F, G$ such that the value function can be written

   $$v(k, c_{-1}) = E + F \log k + G \log c_{-1},$$

   and derive the policy rules.

## B. Growth due to specialization with fringe competition

A representative household orders streams of consumption according to

$$U = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}, \quad \beta \in (0,1), \gamma > 0.$$

The households provide work and lend resources to firms. Their budget constraint is therefore

$$c_t + a_{t+1} \leq (1+r_t)a_t + w_t n_t,$$

where $a_t$ are the amount of assets held by the households.

The consumption good is manufactured competitively from intermediate goods and labor according to

$$y_t = n_t^{1-\alpha} \int_0^{a_t} y_{it}^{\alpha} di.$$

New varieties can be invented by investing $\kappa$ units of the final good in research and development (R&D), i.e.

$$a_{t+1} = a_t + \kappa z_t,$$

where $z_t$ is the amount of output dedicated to R&D.

The intermediate goods are produced using the final good. The original inventor of the good (the monopolist) can use $\psi$ units of final good to produce one unit of intermediate good. Others, called "fringe producers" can produce one unit of the intermediate good using $\nu\psi$ units of the final good, with $1 < \nu < \alpha^{-1}$.

1. Derive the demand curve for intermediate good $i$ by the final goods producers.

2. Ignoring the fringe producers, find the profit maximizing price that the monopolists would choose. Show that this price is higher than the marginal cost of the fringe producers.

3. Based on this result, assume that in equilibrium the price of the intermediate good is equal to the marginal cost of the fringe producers, and that the the entire demand is produced by the monopolists. Derive an expression for the interest rate.

4. How is the growth rate of this economy affected by the fringe competition? In particular, how does it depend on $\nu$? What is the intuition behind this?

## C. An RBC Model with Government Spending

Consider a version of the RBC model where the household's utility is given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\log c_t - \psi n_t), \quad \beta \in (0,1), \psi > 0.$$

The technology is deterministic: $y_t = k_t^{\alpha} n_t^{1-\alpha}$. Capital fully depreciates at the end of each period, i.e. $\delta = 1$.

Assume that there is a government in this economy that must spend $g_t$ units of output at each period, where $g_t$ is a Markov process (think of this as war spending). The resource constraint is therefore

$$c_t + g_t + k_{t+1} \leq y_t.$$

1. Solving the planner's problem, write down the intra-temporal equation, and the Euler inter-temporal equation.
2. Define $\theta_t = y_t/c_t$, and $x_t = g_t/y_t$. Show that the three equations of the model can be combined to one difference equation in $\theta$ and $x$.
3. Suppose that $x_t$ fluctuates around some value $x^*$. Does a steady-state exist for this economy? Is it stable? (the answer may depend on $x^*$).

# Answers

# A. Balanced growth with consumption habits

1.
$$
\begin{aligned}
& v(k,c_{-1}) = \max_{k',c} \Big[ \log c + \gamma \log c_{-1} + \beta v(k',c) \Big] \\
& \text{s.t.} \quad c + k' \le Ak^{\alpha}.
\end{aligned}
$$

2. Define the lagrangian
$$
L = \log c + \gamma \log c_{-1} + \beta v(k',c) - \lambda[c + k' - Ak^{\alpha}].
$$
The FOCs combine to give
$$
\begin{aligned}
& \frac{1}{c} + \beta v_2(k',c) = \lambda, \qquad \beta v_1(k',c) = \lambda, \\
& \frac{1}{c} + \beta v_2(k',c) = \beta v_1(k',c).
\end{aligned}
$$
The envelope equations are
$$
\begin{aligned}
& v_1(k,c_{-1}) = \lambda \alpha A k^{\alpha-1} = \beta \alpha A k^{\alpha-1} v_1(k',c), \\
& v_2(k,c_{-1}) = \frac{\gamma}{c_{-1}}.
\end{aligned}
$$

3. The system we found is
$$
\begin{aligned}
& \frac{1}{c} + \beta v_2(k',c) = \beta v_1(k',c), \\
& v_1(k,c_{-1}) = \beta \alpha A k^{\alpha-1} v_1(k',c), \\
& v_2(k,c_{-1}) = \frac{\gamma}{c_{-1}}.
\end{aligned}
$$
Guess that
$$
v(k,c_{-1}) = E + F \log k + G \log c_{-1},
$$
We have
$$
\begin{aligned}
& \frac{1+\beta G}{c} = \frac{\beta F}{k'}, \\
& \frac{F}{k} = \beta \alpha A k^{\alpha-1} \frac{F}{k'}, \\
& \frac{G}{c_{-1}} = \frac{\gamma}{c_{-1}}.
\end{aligned}
$$

The third equation implies that $G = \gamma$. The second equation gives us the policy rule $k' = \beta\alpha Ak^\alpha$. The first equation is then

$$c = \frac{1}{F}(1+\beta\gamma)\alpha Ak^\alpha.$$

Using the constraint $c + k' = Ak^\alpha$,

$$\begin{aligned}
&\frac{1}{F}(1+\beta\gamma)\alpha Ak^\alpha + \beta\alpha Ak^\alpha = Ak^\alpha \\
&\frac{1}{F}(1+\beta\gamma)\alpha + \beta\alpha = 1 \\
&F = \frac{(1+\beta\gamma)\alpha}{1-\beta\alpha}.
\end{aligned}$$

So

$$c = (1-\beta\alpha)Ak^\alpha.$$

Finally, by substituting the optimal policy into the recursive equation

$$v(k, c_{-1}) = \log[(1-\beta\alpha)Ak^\alpha] + \gamma \log c_{-1} + \beta v\left(\beta\alpha Ak^\alpha, (1-\beta\alpha)Ak^\alpha\right).$$

The left-hand-side is

$$E + \frac{(1+\beta\gamma)\alpha}{1-\beta\alpha}\log k + \gamma \log c_{-1}$$

The right-hand side is

$$\begin{aligned}
&(1+\beta\gamma)\log[(1-\beta\alpha)Ak^\alpha] + \gamma\log c_{-1} + \beta E + \beta\frac{(1+\beta\gamma)\alpha}{1-\beta\alpha}\log(\beta\alpha Ak^\alpha) = \\
&= (1+\beta\gamma)\log[(1-\beta\alpha)A] + \beta\frac{(1+\beta\gamma)\alpha}{1-\beta\alpha}\log(\beta\alpha A) + \beta E + \\
&\qquad + \frac{1+\beta\gamma}{1-\beta\alpha}\alpha\log k + \gamma\log c_{-1}
\end{aligned}$$

Therefore $E$ is determined by

$$\begin{aligned}
&E = (1+\beta\gamma)\log[(1-\beta\alpha)A] + \beta\frac{(1+\beta\gamma)\alpha}{1-\beta\alpha}\log(\beta\alpha A) + \beta E \\
&E = \frac{1+\beta\gamma}{1-\beta}\left[\log[(1-\beta\alpha)A] + \frac{\beta\alpha}{1-\beta\alpha}\log(\beta\alpha A)\right].
\end{aligned}$$

# B. Growth due to specialization with fringe competition

1. The problem of the final good producers is

$$\max_{n_t, y_{it}} \left[ n_t^{1-\alpha} \int_0^{a_t} y_{it}^{\alpha} di - \int_0^{a_t} p_{it} y_{it} - w_t n_t \right].$$

The FOCs

$$\alpha n_t^{1-\alpha} y_{it}^{\alpha-1} = p_{it},$$
$$(1-\alpha) n_t^{-\alpha} \int_0^{a_t} y_{it}^{\alpha} di = w_t.$$

Substituting $n_t = 1$ gives the inverse demand function

$$p_{it} = \alpha y_{it}^{\alpha-1}.$$

2. The intermediate goods producer's problem is

$$\max_{y_{it}} [p_{it} y_{it} - \psi y_{it}] = \max_{y_{it}} [\alpha y_{it}^{\alpha} - \psi y_{it}].$$

The FOC is

$$\alpha^2 y_{it}^{\alpha-1} = \psi \qquad \Rightarrow \qquad y_{it} = \left(\frac{\alpha^2}{\psi}\right)^{1/(1-\alpha)}$$

The monopoly price is therefore

$$p_{it} = \alpha y_{it}^{\alpha-1} = \frac{\psi}{\alpha} > \nu\psi.$$

3. In the equilibrium described, the demand for good $i$ is

$$y_{it} = \left(\frac{\alpha}{p_{it}}\right)^{1/(1-\alpha)} = \left(\frac{\alpha}{\nu\psi}\right)^{1/(1-\alpha)}$$

The monopolist's profits are therefore

$$\pi_{it} = \nu\psi y_{it} - \psi y_{it} = (\nu - 1)\psi \left(\frac{\alpha}{\nu\psi}\right)^{1/(1-\alpha)}$$

This determines the interest rate through

$$\kappa = \sum_{\tau=1}^{\infty} \frac{1}{(1+r)^{\tau}} \pi_{i,t+\tau} = \frac{1}{r}(\nu - 1)\psi \left(\frac{\alpha}{\nu\psi}\right)^{1/(1-\alpha)}$$
$$r = \frac{1}{\kappa}(\nu - 1)\psi \left(\frac{\alpha}{\nu\psi}\right)^{1/(1-\alpha)}$$

4. The household's Euler equation is

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta(1+r) = \frac{\beta}{\kappa}(\nu-1)\psi\left(\frac{\alpha}{\nu\psi}\right)^{1/(1-\alpha)}$$

The rate of growth is therefore the right-hand-side to the power $1/\gamma$.

Notice that

$$\frac{\partial}{\partial\nu}\left[(\nu-1)\nu^{1/(1-\alpha)}\right] = [(2-\alpha)\nu-1]\frac{\nu^{\alpha/(1-\alpha)}}{1-\alpha} > 0,$$

so the growth rate is increasing in $\nu$. This is because competition makes innovation less profitable, and there decreases the growth rate.

# C. An RBC Model with Government Spending

1. We get

$$\psi c = (1-\alpha)\left(\frac{k}{n}\right)^{\alpha},$$

$$\frac{1}{c} = \beta\alpha\mathbb{E}_t\left[\frac{1}{c'}\left(\frac{k'}{n'}\right)^{\alpha-1}\right].$$

2. Substituting $1/c = \theta/y$ into the Euler equation

$$\frac{\theta}{y} = \beta\alpha\mathbb{E}_t\left[\frac{\theta'}{y'}\left(\frac{k'}{n'}\right)^{\alpha-1}\right] = \beta\alpha\mathbb{E}_t\left[\frac{\theta'}{k'^{\alpha}n'^{1-\alpha}}\left(\frac{k'}{n'}\right)^{\alpha-1}\right] = \beta\alpha\mathbb{E}_t\left[\frac{\theta'}{k'}\right].$$

Thus

$$\frac{k'}{y}\theta = \beta\alpha\mathbb{E}_t\left[\theta'\right].$$

Dividing the resource constraint by $y$

$$\frac{c}{y} + \frac{g}{y} + \frac{k'}{y} = 1$$

$$\frac{k'}{y} = 1 - \theta^{-1} - x.$$

So finally

$$(1-\theta^{-1}-x)\theta = \beta\alpha\mathbb{E}_t\left[\theta'\right]$$

$$(1-x)\theta - 1 = \beta\alpha\mathbb{E}_t\left[\theta'\right].$$

3. Searching for a steady-state value $\theta^*$:

$$\theta^* = \frac{1}{1 - x^* - \beta\alpha},$$

so we have a steady state whenever $x^* < 1 - \beta\alpha$. Stability requires

$$\frac{1 - x^*}{\beta\alpha} \leq 1 \Rightarrow x^* \geq 1 - \beta\alpha.$$

Therefore, whenever we have a steady state, it is not stable.