Dixit-Stiglitz Math

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1 Monopolistic Competition

I want to introduce a tool that is frequently used in macroeconomics: the Dixit-Stiglitz aggregator. This tool is used in models where one wants to allow firms to have some market power, but still keep a large number of firms. The idea is that the final good that consumers derive utility from is produced from a continuum of intermediate goods $x(i)$, $i \in [0,1]$. The technology to transform intermediate goods into the final good is

$$Y = \left(\int_0^1 x(i)^\rho di \right)^{1/\rho}.$$

There is a large number of firms that produce the final good using intermediate goods (and, in this example, nothing else), but for each i there is a single monopolistic firm producing $x(i)$. Thus, the final good producers maximize profits taking the price of the intermediary goods as given. Fixing the price of the final good at unity and denoting the price of good i as $p(i)$, the representative final good producer maximizes

$$\pi = Y - \int_0^1 p(i)x(i)di = \left(\int_0^1 x(i)^\rho di \right)^{1/\rho} - \int_0^1 p(i)x(i)di,$$

therefore the demand for $x(i)$ is given by

$$0 = \frac{\partial \pi}{\partial x(i)} = \left(\int_0^1 x(i)^\rho di \right)^{1/\rho - 1} x(i)^{\rho-1} - p(i) = Y^{1-\rho} x(i)^{\rho-1} - p(i). \tag{1}$$

This is perhaps easier to understand by replacing the integral $\int_0^1 di$ with a sum $\sum_{i=1}^N$: the last equation will remain exactly the same in the finite case. Notice that the above defines a demand schedule for the firm that manufactures $x(i)$, taking Y as given.

The intermediate good producers are monopolists each in their product, and use capital and labor to produce with some production function $F(k,n)$, which has all the properties we assumed in previous lectures. Their program is therefore

$$\begin{aligned}
&\max_{k(i),n(i)} [p(i)x(i) - rk(i) - wn(i)] = \\
&= \max_{k(i),n(i)} \left[Y^{1-\rho}x(i)^\rho - rk(i) - wn(i)\right] = \\
&= \max_{k(i),n(i)} \left[Y^{1-\rho}F(k(i),n(i))^\rho - rk(i) - wn(i)\right].
\end{aligned}$$

The first order conditions are

$$r = \rho Y^{1-\rho}F(k(i),n(i))^{\rho-1}F_k(k(i),n(i)) = \rho Y^{1-\rho}x(i)^{\rho-1}F_k(k(i),n(i)), \tag{2a}$$
$$w = \rho Y^{1-\rho}F(k(i),n(i))^{\rho-1}F_n(k(i),n(i)) = \rho Y^{1-\rho}x(i)^{\rho-1}F_n(k(i),n(i)). \tag{2b}$$

We must require $\rho \leq 1$ for these to be sufficient conditions. Equations (2) are a system of two equations that determine $k(i)$ and $n(i)$ (and therefore $x(i)$) for each firm. For a typical production function this system has only one solution, and since r, w, ρ and Y are the same for the entire economy, it follows that all firms will make the same decisions, i.e. $k(i) = k$ and $n(i) = n$ for all i. This also implies that $x(i) = x$ for all i, and therefore

$$Y = \left(\int_0^1 x(i)^\rho di\right)^{1/\rho} = x.$$

Also notice that $p(i) = Y^{1-\rho}x(i)^{\rho-1} = 1$.

Using this in (2) gives

$$\begin{aligned}
r &= \rho F_k(k,n), \\
w &= \rho F_n(k,n).
\end{aligned}$$

Finally, unlike the competitive case, the intermediate good producers are making a profit: using Euler's theorem and (1)

$$\begin{aligned}
\pi(i) = p(i)x(i) - rk(i) - wn(i) = F(k,n) - \rho F_k(k,n)k - \rho F(k,n)n = \\
= (1-\rho)F(k,n) > 0.
\end{aligned}$$

2 The neoclassical growth model with monopolistic power

We can easily embed this into the neoclassical growth framework. Simply replacing the production sector with the above implies the following three equations:

$$
\begin{aligned}
r &= \rho F_k(k,n),\\
w &= \rho F_n(k,n),\\
\pi &= (1-\rho)F(k,n).
\end{aligned}
$$

The households are unchanged, so their first order condition and constraint are still

$$
\begin{aligned}
\frac{u'(c_t)}{u'(c_{t+1})} &= \beta(1-\delta+r_{t+1}),\\
c_t + k_{t+1} &= (1-\delta+r_t)k_t + w_t n_t + \pi_t.
\end{aligned}
$$

Combining the firms equations into the households budget constraint and using Euler's theorem for homogenous functions

$$
\begin{aligned}
c_t + k_{t+1} &= (1-\delta)k_t + \rho F_k(k_t,n_t)k_t + \rho F_n(k_t,n_t)n_t + (1-\rho)F(k_t,n_t) =\\
&= (1-\delta)k_t + \rho F(k_t,n_t) + (1-\rho)F(k_t,n_t) = (1-\delta)k_t + F(k_t,n_t),
\end{aligned}
$$

which is the same resource constraint that the central planner faces. However, notice that the Euler equation is

$$
\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1-\delta+r_{t+1}) = \beta(1-\delta+\rho F_k(k_{t+1},n_{t+1})),
$$

whereas for the central planner we had

$$
\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1-\delta+F_k(k_{t+1},n_{t+1})).
$$

The factor of ρ in front of the marginal product of capital is the manifestation of the inefficiency in this economy. Since firms are monopolistic, they produce too little and capital is underutilized. The steady state k^{SS} in this economy is given by

$$
f'(k^{SS}) = \frac{1}{\rho}\left(\frac{1-\beta}{\beta}+\delta\right) > \frac{1-\beta}{\beta}+\delta = f'(k^*).
$$

Which implies that the steady-state level of capital (and therefore output and consumption) is lower then efficient level k^*.